P R E D I C T A B I L I T Y

A N D

CLIMATE VARIATION ILLUSTRATED BY A LOW-ORDER SYSTEM

B Y

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1. Introduction

The study of atmospheric predictability has mostly been concerned with the uncertainty in the initital state. Indeed, the typical experiment concerned with an aspect of the predictability problem consists of varying the initial state in some manner followed by an investigation of for how long a time the two integrations stay close to each other as measured by a suitable statistical measure of the deviation of one state from another. It has been recognized for some time that in addition to the uncertainty in the initial state it is also necessary to incorporate into the study of predictability the uncertainty in the external forcing of the flow either in the prescription of the heating or the formulation of the various frictional forces (Fleming, 1971).

The role of the external forcing is naturally also important for the predictability because the frictional forces and thereby the dissipation of kinetic energy determines to a large extent the shape of the kinetic energy spectrum which in turn has a bearing on the predictability limit. These factors have been recognized by Lorenz (1969) and others.

The connection between the predictability problem in the general area of weather prediction considered as an initial value problem and the predictability of climate and climate change needs clarification as pointed out in the recent plans for the studies of these problems as part of the Global Atmospheric Research Programme (1974), In view of all the factors mentioned above it is desirable to study simple representations of atmospheric flow into which we can incorporate the uncertainities in the initial state and external forcing. The present paper which deals with the analysis of such a grossly oversimplified system will show that at least a simple system can have several well defined steady states which corresponds to different climatic states. A steady state, i.e. a solution of the governing non-linear equations for which the time derivatives vanish, is not necessarily also an asymptotic state. It may happen that a theoretically possible steady state cannot be reached by the systemfrom any initial state because the steady state is "unstable" in the sense that if the system comes close to the steady state it will - depending on the forcing of the system - move away and approach another steady state which may or may not be stable. In the first case the system will eventually either approach the steady state asymptotically or at least remain in the neighbourhood of the "stable" steady state around which it will oscillate.

Mechanical systemswhich behave in this manner are of course well known. The most simple example is probably the behaviour of a rod which can turn around a horizontal axis in the earth's gravity field. Two steady states are possible each one characterized by a vertical position of the bar.

However, the vertical position where the centre of mass is above the position of the horizontal axis of rotation is of course unstable, while the other is stable. We shall show that our simple analogue of the atmosphere behaves in an analogous fashion.

Much knowledge about the behaviour of the atmosphere has been gained by studies of greatly simplified models. One may in this regard refer to the numerous studies of the so-called advection equation by Platzman (1954), Phillips (1960) and many other investigators (see Platzman (1964) and Benton and Platzman (1972)). The present study is as a matter of fact to be considered as a generalization of the above studies to cases which include a simple forcing mechanism. Most of the studies mentioned above consider either the advection equation without forcing and dissipation or the so-called Burgers equation which in addition contains a dissiptation term of the eddy viscosity type. In the present study we shall replace the eddy viscosity term by the type of term which normally appears in the barotropic vorticity equation when the effects of the surface skin friction is incorporated. The external forcing which in a generalized sense must correspond to the atmospheric heating cannot in our simple model be approximated in a physically realistic fashion. Instead, we shall postulate a Newtonian type of forcing.

The analysis of the simple system considered here can in view of the severe restrictions be considered as nothing more than an illustration of the behaviour of a system which in a limited sense is similar to an atmospheric system.

We shall start the present analysis by considering some low-order examples of systems without forcing and dissipation. In a later section we consider the more general systems.

2. Low Order Examples of the Advection Equation

We consider the equation

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = 0 \tag{2.1}$$

The general solution to the equation has been discussed extensively by Platzman (1964). We shall first consider the representation of (2.1) in the spectral domain. Let us consider an internal $0 \leq x \leq L$ and let the boundary conditions be $U = 0$ at $x = 0$ and $x = L$.

If $U(x,t)$ is represented by the Fourier series

$$U = \sum_{n=1}^{\infty} U(n,t) \sin(nkx), \quad k = \pi / L \tag{2.2}$$

we get the following general set of spectral equations

$$\begin{aligned} \frac{dU(n)}{dt} = & \frac{k}{2} \sum_{m=1}^{\infty} (m+n)\, U(m)\, U(m+n) \\ & - \frac{k}{2} \sum_{m=1}^{n-1} (n-m)\, U(m)\, U(n-m) \\ & - \frac{k}{2} \sum_{m=n+1}^{\infty} (m-n)\, U(m)\, U(m-n) \end{aligned} \tag{2.3}$$

It can be shown that (2.3) is identical to Platzman's spectral equation (Platzman, 1964). The equations (2.3) can be non-dimensionalized by introducing

$$u(n) = \frac{U(n)}{(2E_0)^{1/2}} \quad , \quad \tau = (2E_0)^{\frac{1}{2}} k t \tag{2.4}$$

where

$$2E_0 = \sum_{n=1}^{\infty} U(n)^2 \tag{2.5}$$

twice the kinetic energy, is a conservative quantity for the system. We get:

$$\begin{aligned} \frac{du(n)}{d\tau} = \frac{1}{2} \sum_{m=1}^{\infty} (m+n)\, u(m)\, u(m+n) \\ - \frac{1}{2} \sum_{m=1}^{n-1} (n-m)\, u(m)\, u(n-m) \\ - \frac{1}{2} \sum_{m=n+1}^{\infty} (m-n)\, u(m)\, u(m-n) \end{aligned} \tag{2.6}$$

Considering first the two component system we find with x = u(1) and y = u(2)

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{1}{2} x y \\ \frac{dy}{d\tau} &= -\frac{1}{2} x^2 \end{aligned} \tag{2.7}$$

with the relation $x^2 + y^2 = 1$ from the conservation of kinetic energy. Writing the second equation in (2.7) in the form

$$\frac{dy}{d\tau} = -\frac{1}{2}(1-y^2) \tag{2.8}$$

we may integrate directly and obtain

$$x = -\frac{x_0}{y_0 \sinh(\frac{1}{2}\tau) - \cosh(\frac{1}{2}\tau)}$$

$$y = -\frac{y_0 - \tanh(\frac{1}{2}\tau)}{y_0 \tanh(\frac{1}{2}\tau) - 1} \tag{2.9}$$

An example is shown in figure 1.

Since $\tanh(\frac{1}{2}\tau) = 1$ for $\tau = \infty$ we find that $x = 0$ and $y = 1$ for $\tau = \infty$. In the plane (x,y) we find thus that the asymptotic state is $(0,-1)$, or that the final state is

$$u(x,t) = -\sin(2kx) \tag{2.10}$$

It is clear from (2.7) that the possible steady states are (0,1) and (0, -1). One may indeed ask why the system moves to (0,-1) and not to (0,1). The answer can in this case be obtained directly from the second equation in (2.7) which shows that $dy/d\tau$ is negative and that y therefore must decrease until it reaches the minimum value which is -1. The same

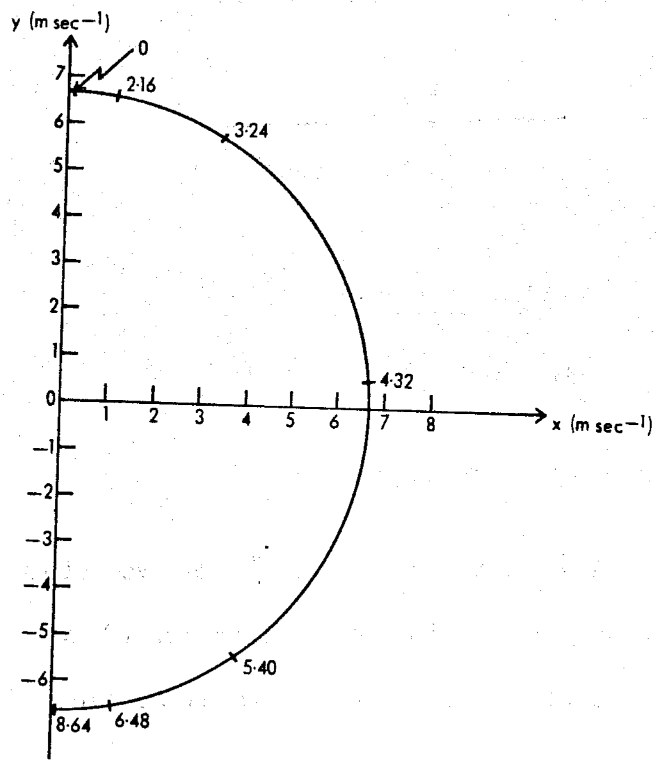


Fig.1: The trajectory of the system (x,y) from the initial condition (0.1, 6.6) in a case of no forcing and no friction. The marks on the curve indicate elapsed time, measured in days.

answer can be obtained by investigating the stability of the steady state (0,1). Using small scale perturbations one obtains the equation

$$\frac{dx'}{d\tau} = \frac{1}{2}x' \tag{2.11}$$

where $x' = x$ and $y' = y-1$. The solution to (2.11) is:

$$x' = x_0' \, e^{\frac{1}{2}\tau} \tag{2.12}$$

showing that x' initially will increase exponentially and that the steady state (0,1) is unstable.

Let us next make an investigation of a three component system. The equations are:

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{1}{2}xy + \frac{1}{2}yz \\ \frac{dy}{d\tau} &= -\frac{1}{2}x^2 + xz \\ \frac{dz}{d\tau} &= -\frac{3}{2}xy \end{aligned} \tag{2.13}$$

We note that $x^2+y^2+z^2 = 1$ for all times with the scaling used here. It is also seen that the only steady states are (0,0,1) and (0,0,-1). Perturbating around anyone of these states we find the perturbation equations

$$\begin{aligned} \frac{dx'}{d\tau} &= \frac{1}{2}z_s y' \\ \frac{dy'}{d\tau} &= z_s x' \\ \frac{dz'}{d\tau} &= 0 \end{aligned} \tag{2.14}$$

where $z_s = \pm 1$. It is seen that the time development of x' (or y') is governed by the equation

$$\frac{d^2 x'}{d\tau^2} - \frac{1}{2} z_s^2 \, x' = 0$$

with solutions of the type

$$x' = C_1 e^{\nu\tau} + C_2 e^{-\nu\tau}, \quad \nu = \left(\tfrac{1}{2} z_s^2\right)^{\frac{1}{2}} \tag{2.15}$$

showing that each of the steady states is unstable.

A similar, but more interesting analysis can be carried out for the four component system. The equations are:

$$\begin{aligned}
\frac{dx}{d\tau} &= \frac{1}{2}(xy + yz + zw) \\
\frac{dy}{d\tau} &= \frac{1}{2}(-x^2 + 2xz + 2yw) \\
\frac{dz}{d\tau} &= \frac{3}{2}(xw - xy) \\
\frac{dw}{d\tau} &= -(2xz + y^2)
\end{aligned} \tag{2.16}$$

The steady states are obtained by setting all the derivatives to zero. Recalling again that $x^2+y^2+z^2+w^2 = 1$ it can be demonstrated that the following steady states are possible

$$\begin{aligned}
&1. \quad x = y = z = 0, \quad w = \pm 1 \\
&2. \quad x = y = w = 0, \quad z = \pm 1 \\
&3. \quad x = x_s, \quad y = x_s, \quad z = -\tfrac{1}{2} x_s, \quad w = x_s \\
&4. \quad x = x_s, \quad y = -x_s, \quad z = -\tfrac{1}{2} x_s, \quad w = -x_s
\end{aligned} \tag{2.17}$$

where x_s in the last expressions is $x_s = \pm 2/(13)^{\frac{1}{2}}$. The interesting point in the four component system is that somewhat more complicated systems can exist. Corresponding to a steady state (x_s, y_s, z_s, w_s) we have the perturbation equations

$$
\begin{aligned}
2\frac{dx'}{d\tau} &= x_s y' + y_s x' + y_s z' + z_s y' + z_s w' + w_s z' \\
2\frac{dy'}{d\tau} &= -2x_s x' + 2x_s z' + 2z_s x' + 2y_s w' + 2w_s y' \\
2\frac{dz'}{d\tau} &= 3x_s w' + 3w_s x' - 3x_s y' - 3y_s x' \\
2\frac{dw'}{d\tau} &= -4x_s z' - 4z_s x' - 4y_s y'
\end{aligned}
\qquad (2.18)
$$

For the solutions of (2.18) we shall assume perturbations of the form $e^{\frac{1}{2}\nu\tau}$. Considering the first steady state we get:

$$
\begin{aligned}
2\frac{dx'}{d\tau} &= w_s z' \\
2\frac{dy'}{d\tau} &= 2w_s y' \\
2\frac{dz'}{d\tau} &= 3w_s x' \\
\frac{dw'}{d\tau} &= 0
\end{aligned}
\qquad (2.19)
$$

The solution for y' is

$$y' = y'_0\, e^{W_s \tau} \qquad (2.20)$$

which will increase or decrease depending on the sign of $w_s(=\pm 1)$. However, the solution for x' and z' are governed by an equation as follows:

$$\frac{d^2 x'}{d\tau^2} - \frac{3}{4} W_s^2\, x' = 0 \qquad (2.21)$$

which has exponential type of solutions of which one will be increasing with time. This steady state is therefore unstable.

Considering next the second type of steady states we find that

$$2\,\frac{dx'}{d\tau} = Z_s\,(y' + w')$$

$$2\,\frac{dy'}{d\tau} = 2\,Z_s\,x'$$

$$\frac{dz'}{d\tau} = 0$$

$$2\,\frac{dw'}{d\tau} = -\,4\,Z_s\,x' \qquad (2.22)$$

The equation governing x' is

$$\frac{d^2 x'}{d\tau^2} + \frac{1}{2} Z_s^2\, x' = 0 \qquad (2.23)$$

which has trigonometric solutions. y' and w' will also have trigonometric solutions, and it follows that $(0,0,\pm 1.0)$ are both stable.

Our next consideration is the third type of steady state where it must be remembered that x_s can be either positive or negative. The perturbation equations are:

$$\frac{dx'}{d\tau} = x_s y' + x_s x' + x_s z' - \tfrac{1}{2} x_s y' - \tfrac{1}{2} x_s w' + x_s z'$$

$$2\frac{dy'}{d\tau} = -2x_s x' + 2x_s z' - x_s x' + 2x_s w' + 2x_s y'$$

$$2\frac{dz'}{d\tau} = 3x_s w' + 3x_s x' - 3x_s y' - 3x_s x'$$

$$2\frac{dw'}{d\tau} = -4x_s z' + 2x_s x' - 4x_s y' \tag{2.24}$$

Introducing the perturbations of the form $e^{\frac{1}{2}\nu\tau}$ we get a set of four linear homogeneous equations. In order to obtain non-trivial solutions the determinant must vanish. Evaluating the determinant and setting $\mu = \nu/x_s$ we find that in addition to $\mu = 0$, μ must also satisfy the equation

$$\mu^3 - 3\mu^2 + \frac{61}{2}\mu - 78 = 0 \tag{2.25}$$

The solutions for μ are:

$$\mu_1 = 2.64 \; ; \quad \mu_{2,3} = 0.18 \pm 5.43i \tag{2.26}$$

Recalling the definition of μ we see that for $x_s > 0$ we will have amplifying solutions while $x_s < 0$ will lead to damped solutions.

A similar analysis can be carried out in the last case. While the technique is the same we find in this case replacing (2.25):

$$\mu^3 + 3\mu^2 + \frac{61}{2}\mu + 78 = 0 \tag{2.27}$$

with the solutions

$$\mu_1 = -2.79; \quad \mu_{2,3} = -0.11 \pm 5.30i \tag{2.28}$$

The steady state considered in case 4 is thus unstable if $x_s < 0$, but stable, if $x_s > 0$.

The main result for the four component system without forcing and dissipation is that non-trivial steady states can be found and that the stability of these states depend upon the sign of x_s.

3. Low Order Systems with Forcing and Dissipation

The general system to be considered in this section is described in the equation:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} = -\varepsilon U + B(U_E - U) \tag{3.1}$$

where the term $-\varepsilon U$ represents the frictional dissipation. We have used this form because it is equivalent to the form simulating the effects of the surface stress in an equivalent barotropic model. In Burgers' equation the term would have the form $\nu \nabla^2 U$ where ν is an eddy viscosity coefficient. The last term in (3.1)is a forcing term of the Newtonian kind. No specific physical process is modelled through the expression, but it is seen that the effect of the term considered in isolation is to increase U if $U < U_E$ and decrease U if $U > U_E$, thus bringing the values of U toward U_E in all cases.

(3.1) can be brought into a non-dimensional form by defining

$$u = \frac{U}{\varepsilon / K}, \quad \tau = \varepsilon t, \quad x = k X, \quad \beta = \frac{B}{\varepsilon}$$

We get:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - u + \beta (u_E - u) \tag{3.2}$$

which means that there is one non-dimensional parameter in the problem.

Using again the series expansion (2.2) we can replace (3.2) by an infinite set of equations of the form:

$$\begin{aligned} \frac{du(n)}{d\tau} = \Big\{ & \frac{1}{2} \sum_{m=1}^{\infty} (m+n)\, u(m)\, u(m+n) \\ & - \frac{1}{2} \sum_{m=1}^{n-1} (n-m)\, u(m)\, u(n-m) \\ & - \frac{1}{2} \sum_{m=n+1}^{\infty} (m-n)\, u(m)\, u(m-n) \Big\} - u(n) + \beta \big(u_E(n) - u(n)\big) \end{aligned} \tag{3.3}$$

where $u_E(m)$ is the Fourier coefficient of the expansion of $u_E(x,t)$.

The system (3.3) may be integrated by numerical methods after a truncation of the series has been adopted. In this section we shall first of all consider the severely truncated system $N = 2$ governed by the equations

$$\frac{dx}{d\tau} = \frac{1}{2}xy - x - \beta(x - x_E)$$
$$\frac{dy}{d\tau} = -\frac{1}{2}x^2 - y - \beta(y - y_E) \qquad (3.4)$$

in which x = u(1), y = u(2), $x_E = u_E(1)$, $y_E = u_E(2)$. If no forcing exists, i.e. $\beta = 0$, we find one steady state, x = 0, y = 0, and we must therefore expect that the system starting from an initial stable (x_o, y_o) will asymptotically approach (0,0). If we were to neglect the non-linear terms we would have the solution

$$x = x_0 e^{-\tau}, \quad y = y_0 e^{-\tau} \qquad (3.5)$$

or

$$r = r_0 e^{-\tau}, \quad r = (x^2 + y^2)^{\frac{1}{2}} \qquad (3.6)$$

The trajectory in the (x,y) plane would thus be a straight line. When the non-linear terms are present it can be expected that the trajectory will deviate considerably from the straight line. However, we notice from (3.4) that we may form the equation

$$\frac{d}{d\tau}(x^2+y^2) = -2(x^2+y^2) \tag{3.7}$$

or

$$\frac{dr^2}{d\tau} = -2r^2 \tag{3.8}$$

giving

$$r^2 = r_0^2 e^{-2\tau} \tag{3.9}$$

or finally

$$r = r_0 e^{-\tau} \tag{3.10}$$

The fact that (3.10) is identical to (3.6) says of course only that the decay of the total kinetic energy is not influenced by the advection terms. The trajectory is however greatly influenced as can be observed in this case because an analytical solution can be found.

It is required to find the solutions to the system

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{1}{2}xy - x \\ \frac{dy}{d\tau} &= -\frac{1}{2}x^2 - y \end{aligned} \tag{3.11}$$

when the initial state is (x_o, y_o) at $\tau = 0$. In spite of the fact that the system is non-linear it turns out that the time-dependent solutions are

$$X = 2A_0 e^{-\tau} \frac{(S^2-1)^{\frac{1}{2}}}{S \cosh(A_0 e^{-\tau}) + \sinh(A_0 e^{-\tau})}$$

$$y = 2A_0 e^{-\tau} \frac{\cosh(A_0 e^{-\tau}) + S \sinh(A_0 e^{-\tau})}{S \cosh(A_0 e^{-\tau}) + \sinh(A_0 e^{-\tau})} \qquad (3.12)$$

where

$$A_0 = \frac{1}{2}\left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \qquad (3.13)$$

and

$$S = \frac{1 - \frac{y_0}{2A_0} \tanh A_0}{\frac{y_0}{2A_0} - \tanh A_0} \qquad (3.14)$$

The details of the solution described by (3.12) to (3.14) are given in the appendix.

Figure 2 shows the trajectory in the (x,y) plane when the initial condition is (10,0), i.e. all the energy is in the first wave number. In the linear problem the trajectory would be from (10,0) to (0,0) along the abscissa. The non-linear solution goes from (10,0) to (0,0) along the curve in Figure 2. It is seen that the energy is transferred from the basis wave number to the second wave number before the dissipation reduces all motion to a state of rest. Even in this simple system there is

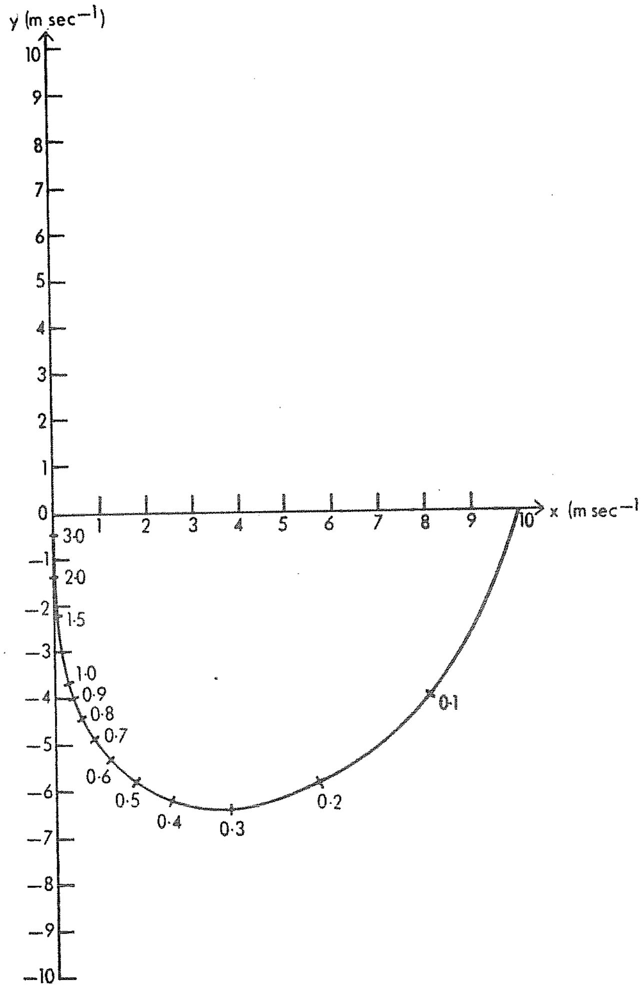


Fig.2: The trajectory of the system (x,y) from the initial condition (10,0) including friction, but no forcing ($\beta = 0$). The marks on the surve indicate values of τ. For $\varepsilon = 3 \times 10^{-6}$ sec^{-1} we find that $\tau = 3$ corresponds approximately to t = 11.5 days.

thus a cascade process from the larger to the smaller scale before the dissipation reduces the motion.

Let us next consider the more general case in which we have both forcing and dissipation. The equations are the system (3.4). In this case there is a possibility for one or several steady states, defined as those states where $dx/d\tau = dy/d\tau = 0$. The steady states may be determined by solving (3.4) under the steady state conditions. Eliminating y from the equations we get the following equation for x

$$x^3 + \left[4(1+\beta)^2 - 2\beta y_E\right] x - 4\beta(1+\beta)x_E = 0 \qquad (3.15)$$

Three real steady states will exist if

$$y_E > 2\frac{(1+\beta)^2}{\beta} + 3 \cdot 2^{-\frac{1}{3}} \beta^{-\frac{1}{3}} (1+\beta)^{\frac{2}{3}} x_E^{\frac{2}{3}} \qquad (3.16)$$

If (3.16) is not satisfied only one steady state will exist. In order to simplify the discussion we shall consider the case in which $x_E = 0$ and $y_E > 2\beta^{-1}(1+\beta)^2$. The steady state values for x are in this case

$$x_{s_1} = 0, \quad x_{s_{2,3}} = \pm\left[2\beta y_E - 4(1+\beta)^2\right]^{\frac{1}{2}} \qquad (3.17)$$

The corresponding values of y are

$$y_{s1} = \frac{\beta}{1+\beta} y_E \; , \quad y_{s2,3} = 2(1+\beta) \tag{3.18}$$

The system may not stay close to anyone of the steady states if disturbed slightly. Since it has not been possible to obtain an analytical solution in this general case one can investigate the various possibilities by numerical procedures. Another possibility is to use perturbation analysis in the neighbourhood of the steady states. Any steady state will be characterised by (x_s, y_s). We consider small deviations, i.e. $x = x_s + x'$, $y = y_s + y'$. Neglecting second order terms as usual we get

$$\begin{aligned} \frac{dx'}{d\tau} &= \frac{1}{2} y_s x' + \frac{1}{2} x_s y' - (1+\beta) x' \\ \frac{dy'}{d\tau} &= - x_s x' \qquad\qquad - (1+\beta) y' \end{aligned} \tag{3.19}$$

In order to study the stability of a given steady state we look for solutions of the type

$$x' = x_0' e^{\nu\tau} \; , \quad y' = y_0' e^{\nu\tau} \tag{3.20}$$

$\nu > 0$ indicates that the steady state is unstable.

The solutions for ν are

$$\nu = \frac{1}{4} y_s - (1+\beta) \pm \frac{1}{2} \left(\frac{1}{4} y_s^2 - 2 x_s^2 \right)^{\frac{1}{2}} \tag{3.21}$$

Corresponding to each of the three steady states we may determine ν , and we find

$$\nu_1 = \begin{cases} \dfrac{\beta y_E}{2(1+\beta)} - (1+\beta) \\ \\ -(1+\beta) \end{cases} \qquad \nu_{2,3} = -\tfrac{1}{2}(1+\beta) \pm \left[\tfrac{9}{4}(1+\beta)^2 - \beta y_E\right]^{\frac{1}{2}} \tag{3.22}$$

Considering the case where three steady states exist, i.e. $\beta y_E - 2(1+\beta)^2 > 0$ we find that the upper value of ν_1 > 0, indicating that the first steady state is unstable, while a closer inspection shows that $\nu_{2,3} < 0$ indicating stability.

As an example we take $x_E = 0$, $y_E = 10$, $\beta = 2$. We have:

$$x_{S1} = 0, \quad y_{S1} = \frac{20}{3}, \quad x_{S2,3} = \pm 2, \quad y_{S2,3} = 6$$

giving

$$\nu_1 = \begin{cases} \frac{1}{3} \\ -3 \end{cases} \qquad \nu_{2,3} = \begin{cases} -1 \\ -2 \end{cases}$$

the instability found above for the steady state $(0, \beta y_E/(1+\beta))$ is due to the advection terms. This is easily seen by disregarding these terms in which case the basic equations are

$$\begin{aligned} \frac{dx}{d\tau} &= -(1+\beta)x \\ \frac{dy}{d\tau} &= -(1+\beta)y + \beta y_E \end{aligned} \tag{3.23}$$

These equations have the steady state solution $(0, \beta y_E/(1+\beta))$ and the general solution

$$x = x_o e^{-(1+\beta)\tau}$$

$$y = y_o e^{-(1+\beta)\tau} + \frac{\beta}{1+\beta} y_E \left(1 - e^{-(1+\beta)\tau}\right) \tag{3.24}$$

which for large values will approach the steady state.

There is of course no guarantee a priori that a system will approach a steady state. Numerical examples presented later will show that a stable steady state will be reached asymptotically in many cases. In addition, the system can approach the stable steady state in several ways. An example of this can be obtained from (3.22). As indicated earlier $\nu_{2,3}$ will be negative if they are real. On the other hand, if

$$\frac{9}{4}(1+\beta)^2 - \beta y_E < 0 \quad (3.25$$

or

$$y_E > \frac{9}{4}\frac{(1+\beta)^2}{\beta} \tag{3.26}$$

$\nu_{2,3}$ will be complex with a negative real part. The steady state will be stable, but close to the point $(x_{s2,3}, y_{s2,3})$ the solution will contain an oscillating part. Using the same numerical example as before we find that no oscillating modes will be present if

$$9 < y_E < \frac{81}{8}$$

why they will be present if

$$y_E > \frac{81}{8}$$

Examples of this behaviour will be shown later.

We shall finally consider the case where only one steady state exists. It will have the coordinates

$$(x_s, y_s) = \left(0, \frac{\beta}{1+\beta} y_E\right)$$

and

$$2\beta y_E - 4(1+\beta)^2 < 0 \qquad (3.27)$$

In this case we find that the single steady state is stable as seen from (3.22) which shows that both values of $\nu_1 < 0$ under these conditons.

The implications of this situation are most interesting. If we select a value of y_E satisfying (3.27) we will have just one steady state which is stable. On the other hand, if we select a slightly different value of y_E satisfying (3.16) with $x_E = 0$ we will have three steady states, and the steady state which was stable for the small value of y_E is now unstable indicating that the system may go to an entirely different steady state. The actual behaviour of the system must be determined by numerical integrations.

4. Some numerical Examples

We shall first show an example which illustrates the predictability problem. Let us compute two trajectories in the (x,y) plane starting from slightly different initial states. In one case we select $x_o = 0.1$ and $y_o = -6.57$ and in the other $x_o = -0.1$ and $y_o = -6.57$. For both calculations we use $x_E = 0$, $y_E = 10$, $\beta = 2$. Figure 3 shows the first trajectory. The second trajectory is symmetrical to the first around the y-axis. We find that each of the two trajectories approach the unstable steady state $(0,6\frac{2}{3})$ for about 500 timesteps. Each timestep is approximately one hour for a suitable value of ε. The two trajectories will thus come closer and closer together for about 20 days. After this time the instability in the neighbourhood of the steady state $(0,6\frac{2}{3})$ takes effect, and one of the trajectories bends to the right very slowly as shown on the figure while the other (not shown) will bend to the left. Notice that it takes about 30 days to travel a very short distance in the (x,y) plane where the trajectory has the greatest curvature. One trajectory will finally end in the stable state (2,6) while the other trajectory ends in the other stable state (-2.6), i.e. in two radically different states. The total time before the stable steady states are reached is about 6000 time-steps or roughly 250 days. The predictability for this simple system is of the order of 3 weeks for the selected case.

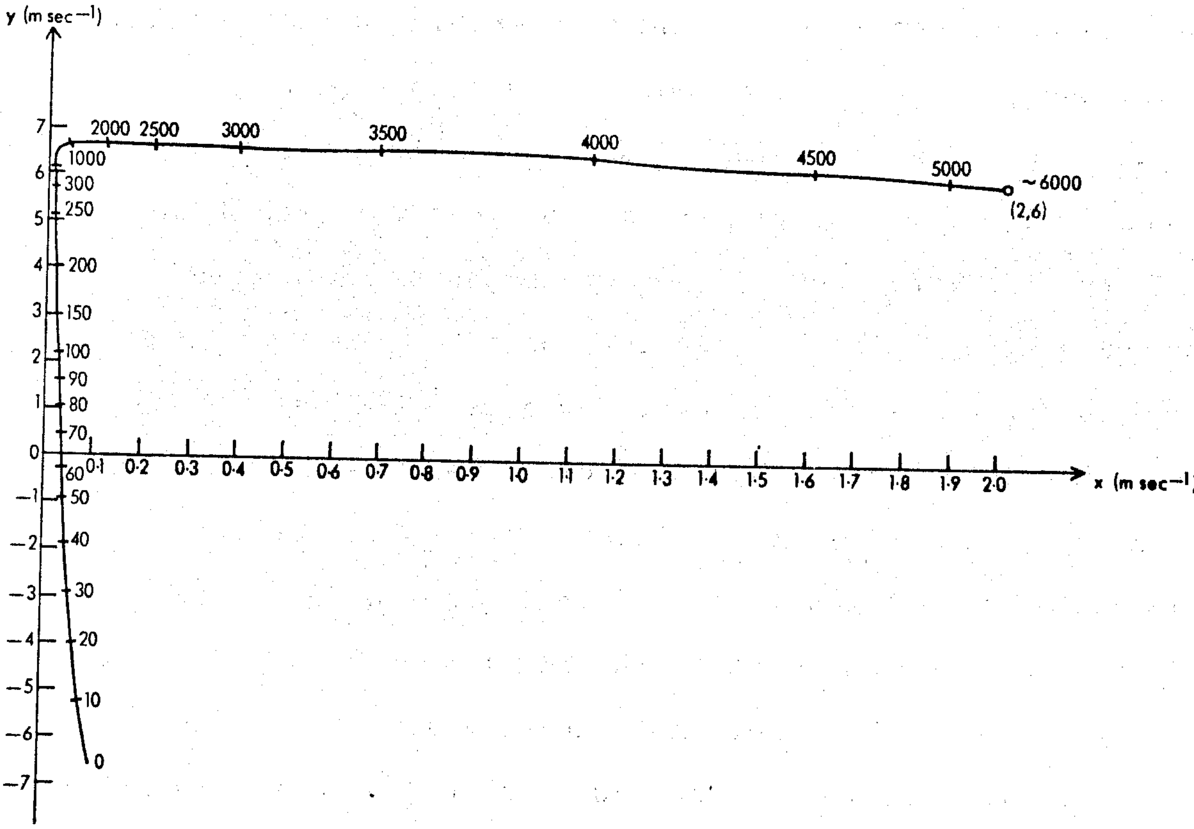


Fig.3: The trajectory of the system (x,y) from the initial condition (0.1,-6.57) with $x_E = 0$, $y_E = 10$ and $\beta = 2$. The possible steady states are (0, 6.67) and (2,6) of which the first is unstable, but the second stable. The marks on the trajectory are numbers of time steps corresponding approximately to elapsed time in hours.

Figure 4 shows a calculation which illustrates a more complicated trajectory. The initial state is (0.1,0). We have selected $x_E = 0$, $y_E = 18$ and $\beta = 2$. According to the linear analysis, see (3.26), we have a case where an oscillating behaviour can be expected in the neighbourhood of the stable steady state (6,6) while the steady state (0,12) is unstable. We find as in the previous case that the system approaches the steady state (0,12). However, as it comes close to this point the instability sets in, and the trajectory curves sharply to the right and comes eventually to the stable steady state (6,6) through a number of oscillations.

The next case illustrated in Figure 5 shows a number of trajectories starting from the initial states: x_o = 0,1, 0.5, 1.0, 1.5,....., 6.0. while $y_o = 0$ for all of them. They have the asymptotic steady state (2,6) using $x_E = 0$, $y_E = 10$ and $\beta = 2$. The cases are similar because they have all the energy in the basic mode (because $y_o = 0$) initially, but the energy level varies very much from one case to another. In spite of this they approach the same steady state (2,6).

As pointed out in connection with (3.22) the steady state $(0, \beta y_E/(1+\beta))$ is unstable if

$$y_E > 2 \frac{(1+\beta)^2}{\beta}$$

and stable if y_E is smaller than this value. The stability of the steady state depends therefore on the intensity of the forcing.

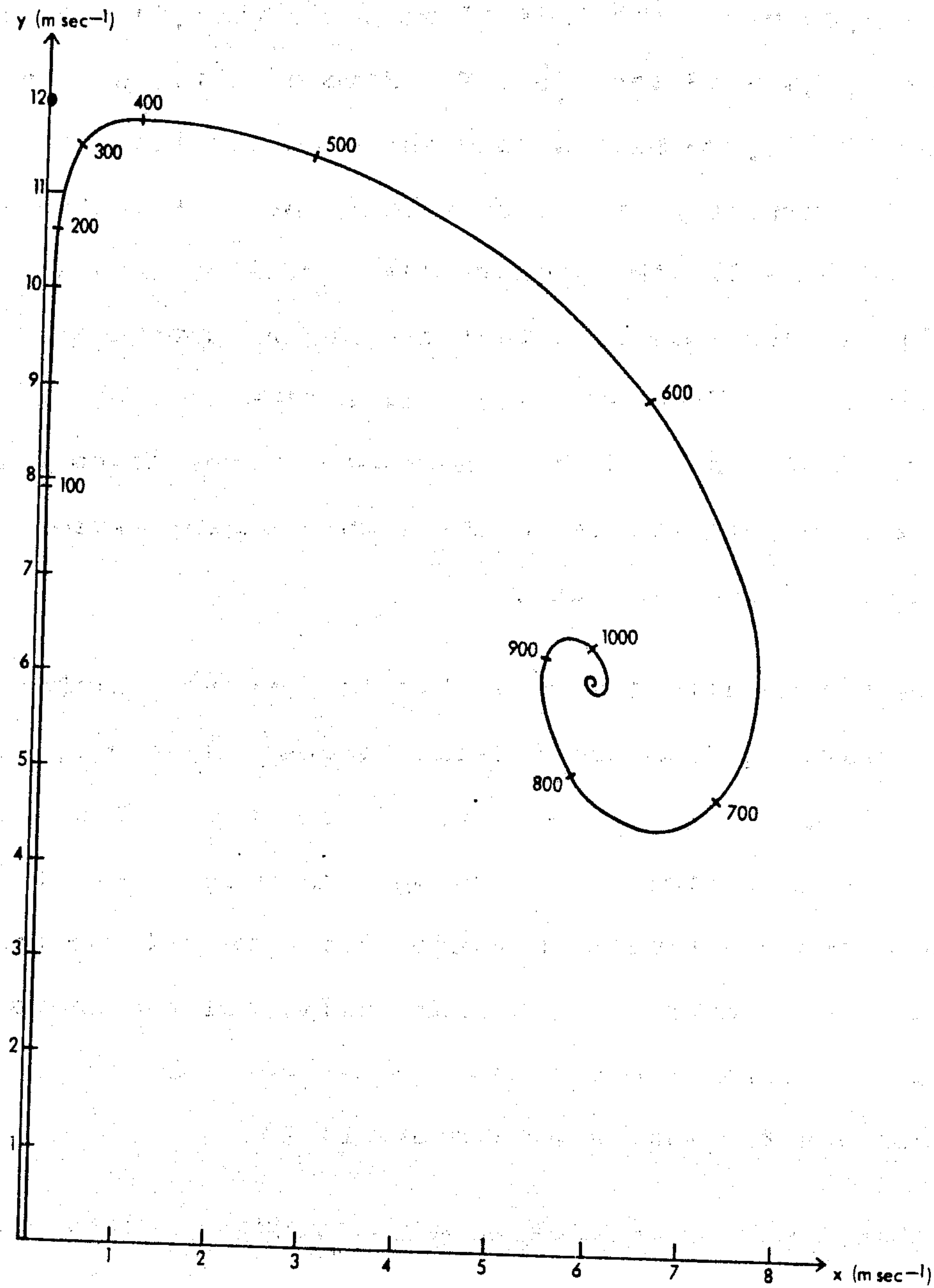


Fig.4: Arrangement as in the previous figure. The initial condition is (0.1, 0) and the values of the parameters are $x_E = 0$, $y_E = 18$, $\beta = 2$. The steady states are (0, 12) and (6,6).

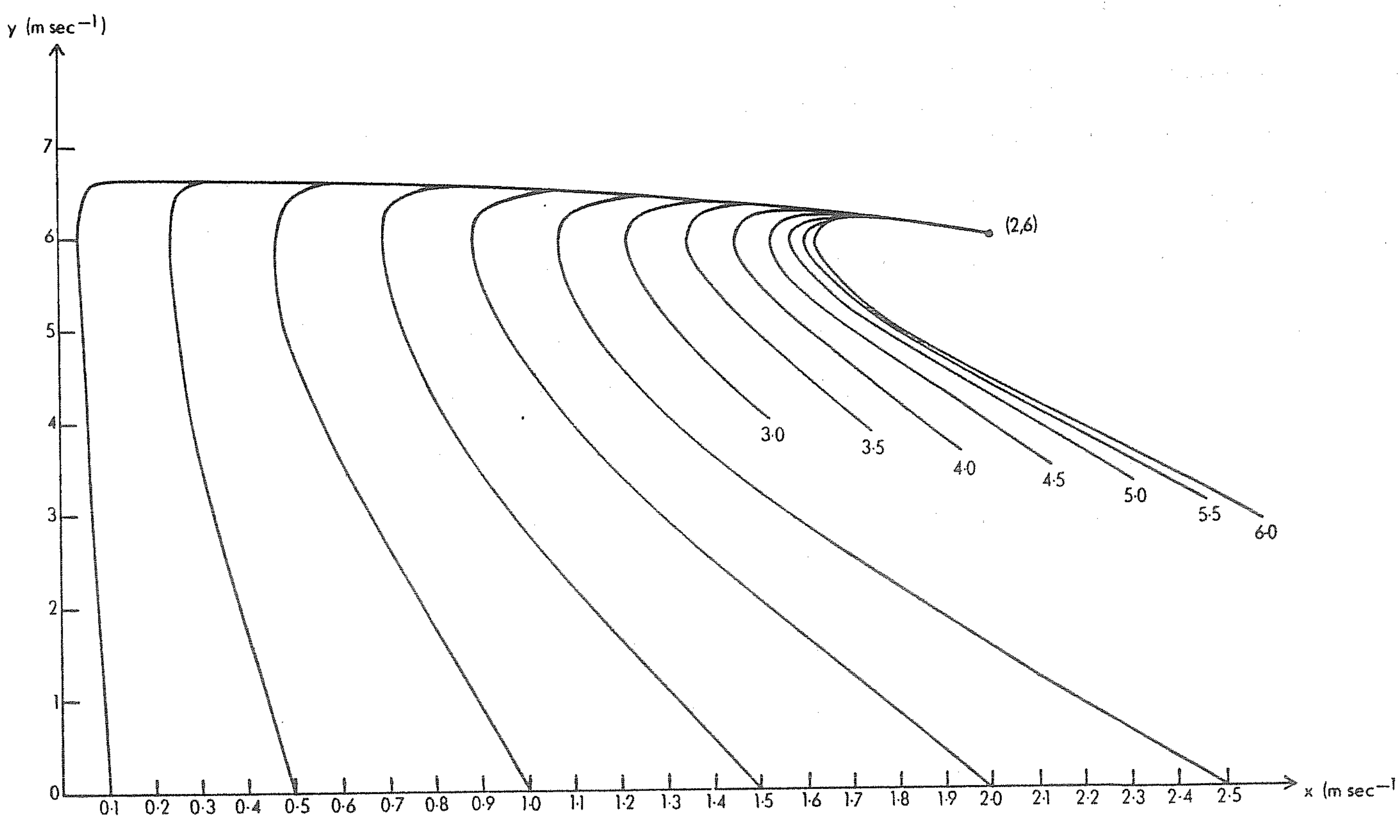


Fig.5: A family of trajectories starting from the x-axis (y=0) in the points 0.1, 0.5, 1.0,1.5 ...., 6.0 and going to the steady state (2,6) . Parameters $x_E = 0$, $y_E = 10$, $\beta = 2$.

Selecting $\beta = 2$ we find that the critical value of Y_E is 9. Two calculations, one with $Y_E = 8.9$ and the other with $Y_E = 9.1$, but with the same initial state and $= 2$ for both calculations, were made in order to illustrate that entirely different "climates" may be obtained due to a slight variation in the forcing, the "heating". The two calculations are illustrated in Figure 6 where the common initial state is (0.1, - 6.0). The calculation with the value $Y_E = 9.1$ (three steady states) approaches first the unstable state (0,5.93). The system stays for a considerable period of time (about 275 days) in the neighbourhood of the unstable steady state, but eventually it moves to the stable steady state (0.63, 6.00) which it reaches after more than a year's time.

The calculations described in Figure 6 are an example of the care which may have to be taken in investigating the sensitivity of the "climate" to variations in the external forcing. If the calculations had been interrupted after one month one would have been tempted to conclude that the effect of a small variation in the external forcing is of smaller importance although as the calculation shows the actual asymptotic states are quite different.

5. Stochastic-dynamic Treatment.

The material covered in the previous sections is based on a deterministic approach. It is naturally also possible to incorporate the initial uncertainties into the problem. We shall again restrict ourselves to the simple two-component system discussed in section 3, i.e. the system described by the two equations in (3.4).

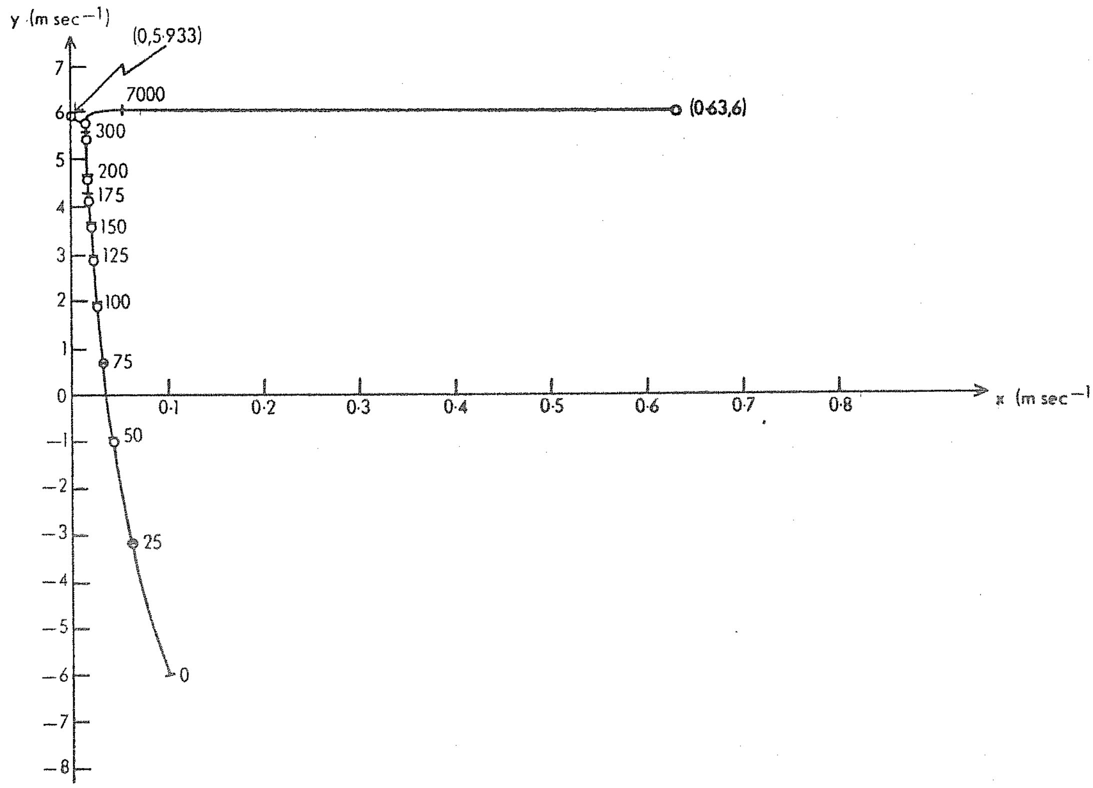


Fig.6: Two trajectories both starting from (0.1, -6) with the parameters $x_E = 0$ and $\beta = 2$. One trajectory marked by circles is computed with $y_E = 8.9$, while the other marked by crossmarks is computed for $y_E = 9.1$. The first trajectory has a stable steady state in the point (0, 5.933), and it arrives there in a stable steady state in the point (0.63, 6), and it arrives there after a very long time.

The first problem is to modify these equations into stochastic-dynamic equations. For this purpose we introduce the estimated value E(x) of a variable x given as a probability distribution. Let us denote

$$\bar{x} = E(x) \qquad (5.1)$$
$$\bar{y} = E(y)$$

In dealing with the system (3.4) it will be necessary to calculate such terms as E(xy). Using the definition of the variance, i.e.

$$\sigma(xy) = E\left[(x-\bar{x})(y-\bar{y})\right] \qquad (5.2)$$

we find

$$E\,(xy) = \bar{x}\,\bar{y} + \sigma(xy) \qquad (5.3)$$

and therefore

$$E(x^2) = \bar{x}^2 + \sigma(x^2) \qquad (5.4)$$
$$E(y^2) = \bar{y}^2 + \sigma(y^2)$$

Assuming that

$$E\left(\frac{dx}{d\tau}\right) = \frac{dE(x)}{d\tau} = \frac{d\bar{x}}{d\tau} \qquad (5.5)$$

we get from (3.4) by applying the operator E

$$\frac{d\bar{x}}{d\tau} = \frac{1}{2}\,\bar{x}\,\bar{y} - (1+\beta)\,\bar{x} + \beta\, x_E + \frac{1}{2}\,\sigma(xy)$$
$$\frac{d\bar{y}}{d\tau} = -\frac{1}{2}\,\bar{x}^2 - (1+\beta)\,\bar{y} + \beta\, y_E - \frac{1}{2}\,\sigma(x^2) \qquad (5.6)$$

Note that in the derivation of (5.6) we have assumed that x_E and y_E are given without any uncertainty. If the system (5.6) is to be useful we need predictive equations for $\sigma(xy)$ and $\sigma(x^2)$. They are most easily derived from the relation

$$\sigma(xy) = E\,(xy) - \overline{x}\,\overline{y} \qquad (5.7)$$

which by differentiation with respect to τ gives

$$\frac{d\,\sigma(xy)}{d\tau} = E\left(x\,\frac{dy}{d\tau} + y\,\frac{dx}{d\tau}\right) - \bar{x}\,\frac{d\bar{y}}{d\tau} - \bar{y}\,\frac{d\bar{x}}{d\tau} \qquad (5.8)$$

$$= E\left[-\tfrac{1}{2}x^3 - (1+\beta)\;xy + \beta\, xy_E + \tfrac{1}{2}\;xy^2 - (1+\beta)\;xy + \beta\, x_E\; y\right]$$

$$+ \tfrac{1}{2}\;\bar{x}^3 + \tfrac{1}{2}\;\bar{x}\,\sigma(x^2) + (1+\beta)\;\;\bar{x}\bar{y} - \beta\,\bar{x}y_E$$

$$- \tfrac{1}{2}\;\bar{x}\bar{y}^2 - \tfrac{1}{2}\bar{y}\,\sigma(xy) + (1+\beta)\;\;\bar{x}\bar{y} - \beta\,\bar{y}\bar{x}_E$$

Upon evaluation we get

$$\frac{d\,\sigma(xy)}{d\tau} = -\;\bar{x}\,\sigma(x^2) + \tfrac{1}{2}\;\bar{x}\;\sigma(y^2) + \tfrac{1}{2}\bar{y}\;\sigma(xy) - 2(1+\beta)\;\sigma(xy) \qquad (5.9)$$

under the very important assumption that we may neglect the third moments. These moments will normally appear in equations for the second moment $\sigma(x^2)$, $\sigma(y^2)$ and $\sigma(xy)$) due to the fact that

$E\,(xyz) = \bar{x}\,\sigma(yz) + \bar{y}\,\sigma(xz) + \bar{z}\,\sigma(xy) + \bar{x}\bar{y}\bar{z} + T\,(xyz)$

where T (xyz) is the third moment. In this preliminary study which will serve as an illustration only we shall use the very simple closure approximation that third moments can be disregarded as presumably small.

Using similar procedures and assumptions we can derive equations for $\sigma(x^2)$ and $\sigma(y^2)$ and we get

$$\frac{d\,\sigma(x^2)}{d\tau} = \bar{x}\,\sigma(xy) + \bar{y}\,\sigma(x^2) - 2(1+\beta)\,\sigma(x^2) \qquad (5.10)$$

and

$$\frac{d\,\sigma(y^2)}{d} = -2\bar{x}\,\sigma(xy) - 2(1+\beta)\,\sigma(y^2) \qquad (5.11)$$

Under our assumptions it is apparent that the equations (5.6),(5.9), (5.10) and (5.11) form a closed system of five equations which can be integrated numerically. However, before we shall describe such calculations we shall as in the previous cases determine the possible steady states of the system and explore the stability of the steady states. It follows without further remarks that the steady states which exist for the deterministic system are also steady states for the stochastic-dynamic system provided $\sigma(x^2) = \sigma(y^2) = \sigma(xy) = 0$. To keep the calculations simple we shall again consider the case where $x_E = 0$. The first of the equations (5.6) may then be written in the following steady state form :

$$\bar{x}\,(\bar{y} - 2\,(1+\beta)) + \sigma(xy) = 0 \tag{5.12}$$

We shall first explore if a solution with $\bar{y} = 2(1+\beta)$, $\sigma(xy) = o$ exists. It is seen that (5.10) is satisfied, and that (5.11) requires $\sigma(y^2) = 0$. It follows then from (5.9) that $\sigma(x^2) = 0$, and we get back to the deterministic solution

$$\bar{x} = \left[2\left(\beta y_E - 2(1+\beta)^2\right]^{\frac{1}{2}},\quad \bar{y} = 2(1+\beta),\quad \sigma(x^2) = \sigma(y^2) = \sigma(xy) = 0$$

The other possibility to satisfy (5.12) with $\sigma(xy) = 0$ is $\bar{x} = 0$.(5.11) says that $\sigma(y^2) = 0$, while (5.10) reduces to

$$\left[\bar{y} - 2(1+\beta)\right]\sigma(x^2) = 0 \tag{5.13}$$

(5.9) is automatically satisfied in this case, and it follows therefore that all steady state equations are satisfied if

$$\bar{x} = 0,\quad \bar{y} = 2(1+\beta),\quad \sigma(x^2) = 2\left[\beta y_E - 2(1+\beta)^2\right]\quad \sigma(y^2) = \sigma(xy) = 0$$

which is a nontrivial solution to the stochastic equations.

After these preliminary cases we may assume that $\bar{y} \neq 2(1+\beta)$. From (5.11) we find

$$\sigma(y^2) = -\frac{\bar{x}}{1+\beta}\,\sigma(xy)$$

$$\sigma(x^2) = -\frac{\bar{x}}{\bar{y}-2(1+\beta)}\,\sigma(xy)$$

Inserting these expressions in the steady state form of (5.9) we find after some calculations

$$\left[\bar{y} - 4(1+\beta)\right]\left[\bar{x}^2 - (1+\beta)(\bar{y} - 2(1+\beta))\right] = c \qquad (5.15)$$

There are two cases to consider. In the first case $\bar{y} = 4(1+\beta)$. Substituting in the system (5.6) we find after elementary calculations that

$$\bar{x} = \pm\left(\beta y_E - 4(1+\beta)^2\right)^{\frac{1}{2}} = \pm Q, \quad \bar{y} = 4(1+\beta) \qquad (5.16)$$

$$\sigma(x^2) = Q^2, \quad \sigma(y^2) - 2Q^2, \quad \sigma(xy) = \mp 2(1+\beta)Q$$

which is a steady state solution.

In the second case we have

$$\bar{y} = \frac{\bar{x}^2}{1+\beta} + 2(1+\beta)$$

in order to satisfy (5.15) . Inserting again in (5.6) we find another steady state solution which may be written as follows

$$\bar{x} = \pm\left(\tfrac{1}{2}\beta y_E - (1+\beta)^2\right)^{\frac{1}{2}} = \pm R, \quad \bar{y} = \frac{R^2}{1+\beta} + 2(1+\beta) \qquad (5.17)$$

$$\sigma(xy) = \mp\frac{R^3}{1+\beta}, \quad \sigma(x^2) = R^2, \quad \sigma(y^2) = \frac{R^4}{(1+\beta)^2}$$

The possible steady state solutions are summarized in the following table

	Case 1	Case 2	Case 3	Case 4	Case 5
$\bar{x}$	$\pm G$	$\pm R$	0	0	$(2\beta y_E - 4(1+\beta)^2$
$\bar{y}$	$4(1+\beta)$	$(1+\beta)^{-1}R^2+2(1+\beta)$	$2(1+\beta)$	$(1+\beta)^{-1}\beta y_E$	$2(1+\beta)$
$\sigma(x^2)$	G^2	R^2	$2(\beta y_E - 2(1+\beta)^2)$	0	0
$\sigma(y^2)$	$2G^2$	$(1+\beta)^{-2}R^4$	0	0	0
$\sigma(xy)$	$\mp 2(1+\beta)G$	$\mp(1+\beta)^{-1}R^3$	0	0	0
	$G=(\beta y_E - 4(1+\beta)^2)^{\frac{1}{2}}$	$R=(\frac{1}{2}\beta y_E - (1+\beta)^2)^{\frac{1}{2}}$			

If anyone of the first three cases shall be valid it must be required that

$$\sigma(xy)^2 \leq \sigma(x^2)\,\sigma(y^2) \tag{5.18}$$

(5.18) leads in case 1 to the condition

$$y_E > 6\frac{(1+\beta)^2}{\beta} \tag{5.19}$$

Case 2 shows that the equality sign in (5.18) applies in all cases, i.e. the correlation coefficient between $\bar{x}$ and $\bar{y}$ is either 1 or -1. For Case 3 (5.18) is satisfied in all cases, but it must naturally be required that

$$y_E > 2\,\frac{(1+\beta)^2}{\beta} \tag{5.20}$$

since $\sigma(x^2)$ must be positive.

The next important question is whether or not the cases 1,2 and 3 represent stable or unstable steady states. The question can be answered by performing a perturbation analysis by linearizing the equations (5.6), (5.9), (5.10) and (5.11) using a steady state as the basis state. Let an arbitrary steady state be denoted by $\bar{x}_s$, $\bar{y}_s$, $\sigma_s(x^2)$, $\sigma_s(y^2)$ and $\sigma_s(xy)$. Using the same form of the perturbation quantities as before, i.e. proportional to exp $(\nu\tau)$, we find that the following determinant must vanish in order to obtain non-trivial solutions, i.e.

$$\begin{vmatrix} \frac{1}{2}\bar{y}_s-(1+\beta)-\nu & \frac{1}{2}\bar{x}_s & \frac{1}{2} & 0 & 0 \\ -\bar{x}_s & -(1+\beta)-\nu & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2}\sigma_s(y^2)-\sigma_s(x^2) & \frac{1}{2}\sigma_s(xy) & \frac{1}{2}\bar{y}_s-2(1+\beta)-\nu & -\bar{x}_s & \frac{1}{2}\bar{x}_s \\ \sigma_s(xy) & \sigma_s(x^2) & \bar{x}_s & \bar{y}_s-2(1+\beta)-\nu & 0 \\ -2\sigma_s(xy) & 0 & -2\bar{x}_s & 0 & -2(1+\beta)-\nu \end{vmatrix} = 0 \quad (5.21)$$

Case 3 is so simple that a direct calculation of (5.21) is straight forward. Substituting the values from Case 3 in (5.21) and evaluating the determinant one gets

$$\nu(1+\beta+\nu)+(\beta y_c-2(1+\beta)^2)=0 \qquad (5.22)$$

giving

$$\nu=-\frac{1}{2}(1+\beta)\pm\left(\frac{9}{4}(1+\beta)^2-\beta y_c\right)^{\frac{1}{2}} \qquad (5.23)$$

If y_E is so large that the radicand is negative we find a negative real part of ν, indicating stability. On the other hand, if the radicand is positive we find that both values of ν are negative if

$$\frac{9}{4}(1+\beta)^2 - \beta y_E < \left(\frac{1+\beta}{2}\right)^2 \qquad (5.24)$$

leading to

$$y_E > 2\,\frac{(1+\beta)^2}{\beta}$$

which is the same inequality as (5.20). The result is therefore that the steady state described by Case 3 is stable whenever it exists, i.e. when (5.20) is satisfied. Referring to the analysis carried out in section 3 it should be noted that Case 3 and Case 4 are intimately connected. Case 4 represents a stable steady state when (5.20) is not satisfied, i.e. when

$$y_E < 2\,\frac{(1+\beta)^2}{\beta}$$

In this case all variances and covariances are zero. For $y_E = 2\,(1+\beta)^2\beta^{-1}$ Case 3 and Case 4 coincide. Finally, when $y_E > 2\,(1+\beta)^2\beta^{-1}$ Case 3 will exist and is a stable steady state.

In Case 1 and 2 numerical methods have been used to determine the eigen-values. For both cases the value $\beta = 2$ was selected. According to (5.19) we must consider values of $y_E > 27$ in Case 1, while Case 2 requires values of $y_E > 9$ for $\bar{x}_S$ to be real. The results of the numerical investigation are that in all cases one positive eigen-value was found for both of the two cases. We must therefore conclude that both Case 1 and Case 2 represent unstable steady states.

The results of the stability investigation are summarized in the following table when we have used $\beta = 2$ in the numerical evaluation

Case 1	Case 2	Case 3	Case 4	Case 5
Exists for $y_E > 27$	Exists for $y_E > 9$	Exists for $y_E > 9$	Exists for all y_E	Exists for $y_E > 9$
Unstable for $y_E > 27$	Unstable for $y_E > 9$	Stable for $y_E > 9$.	Unstable for $y_E > 9$ Stable for $y_E < 9$	Stable for $y_E > 9$

It is seen that for $y_E < 9$ one and only one stable steady state exists. It will have vanishing values of $\sigma(x^2)$, $\sigma(y^2)$ and $\sigma(xy)$. For $y_E > 9$ two stable steady states exist of which Case 3 represents a stochastic-dynamic solution, while Case 5 is a steady state with vanishing values of $\sigma(x^2)$, $\sigma(y^2)$ and $\sigma(xy)$.

6. Energetics of the stochastic-dynamic Model.

The kinetic energy of the first moment is

$$K_1 = \frac{1}{2}\left(\bar{x}^2 + \bar{y}^2\right) \tag{6.1}$$

while the kinetic energy of the second moment may be defined as

$$K_2 = \frac{1}{2}\left(\sigma(x^2) + \sigma(y^2)\right) \tag{6.2}$$

It follows from (5.6) that

$$\frac{dK_1}{d\tau} = -2K_1 - \beta\left[\bar{x}(\bar{x} - x_E) + \bar{y}(\bar{y} - y_E)\right] + \frac{1}{2}\left[\bar{x}\,\sigma(xy) - \bar{y}\,\sigma(x^2)\right] \tag{6.3}$$

while addition of (5.10) and (5.11) followed by a division by 2 leads to

$$\frac{dK_2}{d\tau} = -2K_2 - 2\beta K_2 - \frac{1}{2}\left[\bar{x}\,\sigma(xy) - \bar{y}\,\sigma(x^2)\right] \tag{6.4}$$

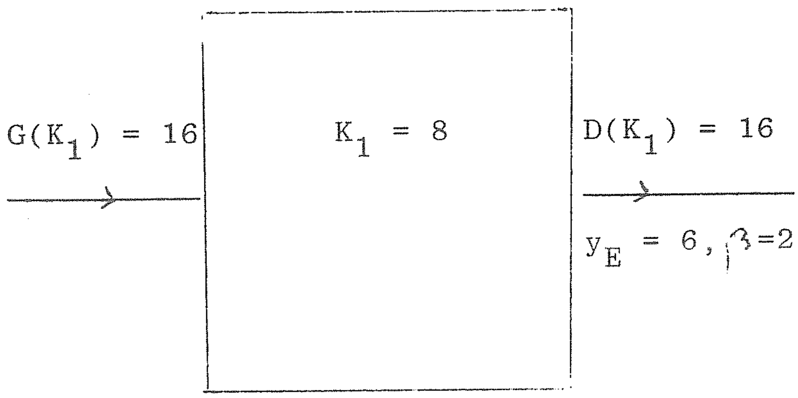
Recalling that β measures the intensity of the Newtonian forcing we may write

$$\begin{aligned}
\frac{dK_1}{d\tau} &= -D(K_1) + G(K_1) - C(K_1, K_2) \\
\frac{dK_2}{d\tau} &= -D(K_2) + G(K_2) + C(K_1, K_2)
\end{aligned} \tag{6.5}$$

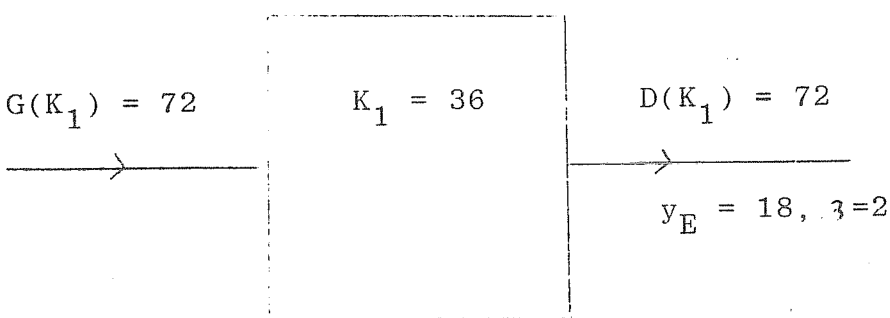
where

$$\begin{aligned}
G(K_1) &= \beta\left[\bar{x}(x_E - \bar{x}) + \bar{y}(y_E - \bar{y})\right] \\
G(K_2) &= -2\beta K_2 \\
D(K_1) &= +2K_1 \\
D(K_2) &= +2K_2 \\
C(K_1, K_2) &= \frac{1}{2}\left[\bar{y}\,\sigma(x^2) - \bar{x}\,\sigma(xy)\right]
\end{aligned} \tag{6.6}$$

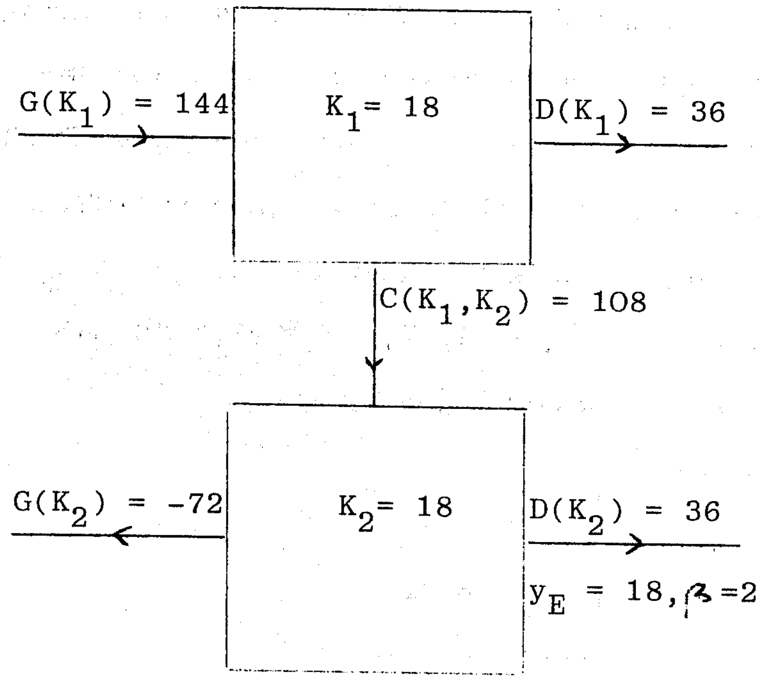
It is seen that $D(K_1)>0$, $D(K_2)>0$, $G(K_2)<0$ while the remaining quantities may have either sign. It is instructive to consider the energy diagrams for the stable steady states. For $y_E<9$ ($\beta=2$) we have one such state (Case 4). In this case we have vanishing variances and covariances and consequently $K_2=0$ and $G(K_2)= D(K_2)=C(K_1,K_2) = 0$. For $x_E=0$, $y_E=6$ we find $\bar{x} = 0$, $\bar{y} = 4$, $K_1= 8$, $G(K_1)=16$ and $D(K_1) = 16$ as illustrated in the following diagram :



On the other hand, for $y_E > 9$ ($\beta=2$) we have two stable steady states which are quite different from an energy point of view (Case 3 and Case 5). Let us first consider Case 5. Selecting $y_E = 18$ we find $\bar{x} = 6$, $\bar{y} = 6$, $K_1 = 36$, $G(K_1) = 72$, $D(K_1) = 72$ as illustrated in the following diagram :



Case 3 gives $\bar{x} = 0$, $\bar{y} = 6$, $\sigma(x^2) = 36$, $\sigma(y^2) = \sigma(xy) = 0$ and therefore $K_1 = 18$, $K_2 = 18$, $G(K_1) = 144$, $D(K_1) = 36$, $C(K_1,K_2) = 108$, $G(K_2) = -72$, $D(K_2) = 36$ as illustrated in the following diagram :



The fact that K_1 and K_2 are equal in the example above is due to the choice of $y_E = 18$. In general we have for this case as seen from the steady state solution for Case 3 that

$$\frac{K_2}{K_1} = \frac{\beta y_E}{2(1+\beta)^2} - 1$$

indicating that K_2 is large relative to K_1, when y_E is large . It should however be noted that the total energies for Case 3 and Case 5 are the same because

$$K_1 + K_2 = \beta y_E$$

for Case 3, while

$$K_1 = \beta y_E \ , \quad K_2 = 0$$

for Case 5.

7. Concluding Remarks

The equation studied in this paper cannot be considered as a model of the atmosphere. It is an equation which contains time-dependence and non-linear effects similar to those appearing in the atmospheric models. In addition, the equation incorporates a forcing and a dissipation.

The study of the properties of the equation shows that the predictability problem can be conveniently illustrated by numerical integration of the equation. The low order system derived from the original differential equation may be used to illustrate the existence of one or several stationary states depending on the external forcing and the value of the only non-dimensional parameter in the problem. Due to the simplicity of the low order system it is furthermore possible to study the stability of the possible stationary states. In this way it is demonstrated that for small values of the forcing a single stationary stable state exists. On the other hand, large values of the forcing leads to the existence of three possible stationary states of which one is unstable and the remaining two stable. A critical value of the forcing exists. Numerical integrations with a slightly subcritical and a slightly supercritical value are performed in order to illustrate the dependence of the stable stationary state on the external forcing.

The low order deterministic system can be generalized to include the dynamic-stochastic approach. The present system has a closure approximation of a very simple nature, i.e. the neglect of third and higher moments. The existence and stability of stationary states are investigated resulting in one stable stationary state for small values of the external forcing. This state is characterized by vanishing variances and covariances. Numerical experiments indicate that the stationary state will be reached regardless of the initial conditions in amplitudes and their uncertainty. On the other hand, for large values of the external forcing two stable stationary states exist. Preliminary numerical studies seem to indicate that from a given initial state, given by the initial position $\bar{x}$, $\bar{y}$ and initial uncertainities $\sigma(x^2)$, $\sigma(y^2)$ and $\sigma(xy)$ the system will eventually arrive in the steady state which is closest to the initial state, but further studies are necessary in order to confirm this preliminary impression.

8. Acknowledgements

The author would like to thank Mr. E. Knighting for very stimulating discussions of several aspects of this paper. Mr. Z. Janjic has programmed and performed all numerical integrations of the various systems considered in the study with great efficiency and competence.

Appendix

This appendix contains the details of the analytical solution of the problem in section 3 in which dissipation, but no forcing is included.

The equations are:

$$\frac{dx}{d\tau} = \frac{1}{2} xy - x \tag{A.1}$$

$$\frac{dy}{d\tau} = -\frac{1}{2} x^2 - y \tag{A.2}$$

Multiplying (A.1) by 2x we find

$$\frac{dx^2}{d\tau} = x^2 (y-2) \tag{A.3}$$

while (A.2) can be written in the form

$$x^2 = -2\left(\frac{dy}{d\tau} + y\right) \tag{A.4}$$

Substitution of (A.4) in (A.3) gives

$$\frac{d^2y}{d\tau^2} - (y-3)\frac{dy}{d\tau} - y(y-2) = 0 \tag{A.5}$$

In (A.5) we introduce the transformation

$$3z = y-3 ; \quad y = 3(z+1) \tag{A.6}$$

giving

$$\frac{d^2z}{d\tau^2} - 3z\frac{dz}{d\tau} - (3z^2 + 4z + 1) = 0 \tag{A.7}$$

which may be written in the form

$$\frac{d}{d\tau}\left[\frac{dz}{d\tau}-\frac{3}{2}z^2-2z\right]+2\left[\frac{dz}{d\tau}-\frac{3}{2}z^2-2z\right]=1 \tag{A.8}$$

Denoting

$$Q=\frac{dz}{d\tau}-\frac{3}{2}z^2-2z \tag{A.9}$$

we find

$$\frac{dQ}{d\tau}+2Q=1 \tag{A.10}$$

which has the solution

$$Q=\frac{1}{2}\left(1+C_o\ e^{-2\tau}\right) \tag{A.11}$$

The integration constant C_o can be determined from the initial conditions. Let $x = x_o$, $y = y_o$ at $\tau = 0$. We have then

$$z_o=\frac{1}{3}(y_o-3) \tag{A.12}$$

and from (A.2) we find

$$\left(\frac{dz}{d\tau}\right)_o=-\frac{1}{6}x_o^2-(z_o+1) \tag{A.13}$$

The initial value of Q, i.e. Q_o, has therefore the value

$$Q_o=-\frac{1}{6}\left(x_o^2+y_o^2\right)+\frac{1}{2} \tag{A.14}$$

On the other hand, (A.11) gives for $\tau = 0$

$$Q_0 = \frac{1}{2}(1 + C_0) \tag{A.15}$$

Combining (A.14) and (A.15) we find

$$C_0 = -\frac{1}{3}(x_0^2 + y_0^2) \tag{A.16}$$

Using (A.16) we find that the equation for z is

$$\frac{dz}{d\tau} - \frac{3}{2}z^2 - 2z = \frac{1}{2} + \frac{1}{2}C_0 e^{-2\tau} \tag{A.17}$$

In order to solve the nonlinear equation (A.17) we introduce the transformations

$$z = -\frac{2}{3}\frac{1}{V}\frac{dV}{d\tau} \tag{A.18}$$

giving

$$\frac{d^2V}{d\tau^2} - 2\frac{dV}{d\tau} + \frac{3}{4}(1 + C_0 e^{-2\tau})V = 0 \tag{A.19}$$

which is a linear equation for V. This equation can be further simplified by introducing

$$\zeta = e^{-2\tau} \tag{A.20}$$

giving

$$\zeta^2 \frac{d^2V}{d\zeta^2} + 2\zeta\frac{dV}{d\zeta} + \frac{3}{16}(C_0\zeta + 1)V = 0 \tag{A.21}$$

(A.21) is one of the standard forms of the differential equation for the Bessel functions. It turns out that the solutions are functions of the order ½ with a purely imaginary agreement. Denoting $A_o = \frac{1}{2}(x_o^2 + y_o^2)^{\frac{1}{2}}$ we find that

$$V = \zeta^{-\frac{3}{4}}\left(K_1 \sinh\left(A_o \zeta^{\frac{1}{2}}\right) + K_2 \cosh\left(A_o \zeta^{\frac{1}{2}}\right)\right) \qquad (A.22)$$

From (A.22) we can evaluate z from (A.18) and then y using (A.6). Expressed in terms of τ, using (A.20), one obtains

$$y = 2A_o e^{-\tau} \frac{\cosh(A_o e^{-\tau}) + S \sinh(A_o e^{-\tau})}{\sinh(A_o e^{-\tau}) + S \cosh(A_o e^{-\tau})} \qquad (A.23)$$

where $S = K_2/K_1$

Since $y = y_o$ for $\tau = 0$ we find that following value for S:

$$S = \frac{1 - \frac{y_o}{2A_o} \tanh A_o}{\frac{y_o}{2A} - \tanh A_o} \qquad (A.24)$$

The solution for x is most easily obtained by noting that a first integral of (A.1) and (A.2) is:

$$x^2 + y^2 = 4A_o^2 e^{-2\tau}$$

The final result is

$$x = 2A_o e^{-\tau} \frac{(S^2 - 1)^{\frac{1}{2}}}{S \cosh(A_o e^{-\tau}) + \sinh(A_o e^{-\tau})} \qquad (A.25)$$

REFERENCES

Benton, E.R. and G.W. Platzman, 1972:
A table of solutions of the one-dimensional Burgers' equation. Quarterly of Applied Mathematics, pp. 195-212.

Fleming, R.J., 1971:
On stochastic-dynamic prediction II predictability and utility, Monthly Weather Review, Vol.99, pp.927-938.

Global Atmospheric Research Programme, 1974:
The physical basis of climate and climate modelling, GARP Publication Series, No. 16, ICSU/WMO, 265 pp.

Lorenz, E.N. 1969:
The predictability of a flow which possesses many scales of motion, Tellus, Vol. 21, pp. 289-307.

Phillips, N.A., 1960:
Numerical Weather Prediction, Advances in Computer, Vol. 1, pp. 43-86, New York Academic Press, 316 pp.

Platzman, G.W., 1964 :
An exact integral of complete spectral equations for unsteady one-dimensional flow, Tellus, Vol. 16, pp. 422-431.