FINITE-DIFFERENCE SCHEMES FOR THE PRESSURE GRADIENT FORCE AND FOR THE HYDOSTATIC EQUATION

Fedor Mesinger*
Department of Meteorology, University of Belgrade, Yugoslavia

and

Zavisa I. Janjić
Federal Hydrometeorological Institute, Belgrade, Yugoslavia

1. INTRODUCTION: PROBLEMS AND GUIDING PRINCIPLES

Since the pressure gradient force involves horizontal differencing of the geopotential, discretization of the pressure gradient force is related to the discretization of the hydrostatic equation. Another term of the primitive equations strongly coupled with the pressure gradient force is the "omega-alpha" term of the thermodynamic equation. It is perhaps generally accepted as very important that this term be defined so as to guarantee that there will be no false energy generation in transformations between the kinetic and the total potential energy. Thus, we shall in this lecture mainly be concerned with the terms/equations

$$\frac{\partial \underline{V}}{\partial t} = \ldots - \nabla_p \phi + \ldots , \qquad (1.1)$$

$$\frac{\partial \phi}{\partial p} = - \frac{RT}{p} , \qquad (1.2)$$

$$\frac{dT}{dt} = \frac{\kappa T \omega}{p} . \qquad (1.3)$$

Here $\underline{V}$ is the horizontal velocity, t is time, p is pressure, ϕ is geopotential, R is the gas constant, T is temperature, d/dt is the material time derivative, κ is R/c_p, where c_p is the specific heat at constant pressure, and $\omega \equiv dp/dt$ is the vertical velocity in the pressure system (used at this point for reasons of convenience). Finally, the subscript of the del operator, as usual, denotes the variable held constant under the differentiation process.

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* Lecture presented by F. Mesinger

Problems and properties of schemes used for the discretization of the considered terms depend on the choice of the vertical coordinate. To avoid difficulties with the lower boundary conditions, in almost all comprehensive atmospheric models terrain-following coordinates are used. A major problem then is related to the appearance of two terms in the expression for the pressure gradient force. For example, with the original sigma coordinate of Phillips (1957)

$$\sigma \equiv p/p_s \ , \tag{1.4}$$

where the subscript s stands for surface values, one obtains

$$-\nabla_p \phi = - \nabla_\sigma \phi - RT \nabla \ln p_s \ . \tag{1.5}$$

Over sloping terrain the two terms on the right hand side of (1.5) tend to be large in absolute values and to have opposite signs. If, say, they are individually ten times greater than their sum, a 1% error in temperature (2-3°C) will result in a 10% error in the pressure gradient force (Sundqvist, 1975)

It has not been proved that it is possible to construct a scheme which would be free of this error, and, for example, give no false pressure gradient force in the simple case of an atmosphere in hydrostatic equilibrium. However, a number of techniques have been designed with the aim of keeping the error within hopefully tolerable limits. In addition, schemes have been constructed which maintain an integral property of the pressure gradient force (Arakawa, 1972). This feature was apparently also largely directed at controlling the effects of the error by putting a constraint on the error in such a way that there are no spurious sources or sinks of the vertically integrated vorticity (Arakawa and Suarez, 1983). We shall in this section briefly review these various approaches.

A technique practiced as an early response to the recognition of the problem was the vertical interpolation of geopotential from sigma back to constant pressure surfaces (Smagorinsky et al., 1967; Kurihara, 1968). Problems remain near the ground however, where extrapolation to obtain subterranean geopotentials is then needed. In addition, resulting schemes may be relatively time consuming. Finally, with this approach it is not obvious how to construct the scheme for the omega-alpha term in such a way as to prevent false energy generation in the energy conversion process.

The Arakawa scheme (Arakawa, 1972; Arakawa and Lamb, 1977) maintained the property of the pressure gradient force that there is no generation of circulation of vertically integrated momentum along a contour of the surface topography. Namely, we have, with g denoting gravity, and subscript T values at the top pressure surface of the model,

$$-\frac{1}{g}\int_{p_T}^{p_s} \nabla_p \phi \, dp = -\frac{1}{g}\left[\nabla \int_{p_T}^{p_s} (\phi - \phi_s) dp + (p_s - p_T)\nabla\phi_s\right] \qquad (1.6)$$

Thus, the integral of the left hand side of (1.6) along any closed curve following a contour of the surface topography will be zero, and no circulation of the vertically integrated momentum along such a curve will be generated by the pressure gradient force. The Arakawa (1972) scheme has been further developed to satisfy additional requirements (Phillips, 1974; Tokioka, 1978), and is being extensively used in general circulation and weather prediction models (e.g., Stackpole, et al., 1980).

A scheme maintaining the one-dimensional analog of the integral of (1.6), necessary from the point of view of the angular momentum conservation, has been developed by Simmons and Burridge (1981). The Simmons and Burridge scheme was however designed in terms of a very general "hybrid" vertical coordinate; and for a "local" hydrostatic equation, thereby avoiding the physically unrealistic dependence of geopotential of the lowest level on the temperature of all model levels that occurs in the Arakawa (1972) scheme. A most concise formulation of the vertical differencing requirement in sigma coordinates resulting in schemes that satisfy an analog of (1.6) has recently been achieved by Arakawa and Suarez (1983). The Arakawa and Suarez formulation permits an arbitrary choice of the hydrostatic equation.

For a given atmosphere in hydrostatic equilibrium, Corby et al., (1972) have defined the difference analog of the pressure gradient (second) term of the pressure gradient force so that it was exactly balancing the geopotential gradient (first) term. It was expected that with no error for this chosen, typical, atmosphere, large errors would be avoided in more general cases. Subsequently, this technique has been used and/or generalized by a number of authors: Nakamura (1978), Sadourny et al., (1981), Simmons and Burridge (1981), and Arakawa and Suarez (1983).

Janjić (1977) addresses the generation of the error in the general case. The basic idea is present also in an earlier paper by Rousseau and Pham (1971). Note that, say with the original definition of the sigma coordinate,

$$- \nabla_p \phi = - \nabla_\sigma \phi - RT \nabla \ln p_s = - \nabla_\sigma \phi + (\nabla_\sigma \phi - \nabla_p \phi) = - \nabla_\sigma \phi + \nabla(\phi_\sigma - \phi_p) .$$

Thus, the pressure gradient term of the pressure gradient force can be considered to represent a hydrostatic correction to the geopotential gradient term. In a difference scheme, it corresponds to a vertical extrapolation/

interpolation of geopotential from the sigma surface, or surfaces, back to the constant pressure surface of the considered velocity point, say the surface p^* in Fig. 1. It was pointed out by Rousseau and Pham that for minimization of the error the formulation of that second term should be "coherent" with that of the first term and with the formulation of the hydrostatic equation. In the paper by Janjić the same requirement is put forth as that for a "hydrostatic consistency" of the scheme. The geopotential gradient term implies a certain difference scheme relating vertical increments of geopotential from pressure to the sigma surface, increments from ⊕ to x points in Fig. 1, to some grid point values of temperature. If a different scheme and/or different grid point values are used to evaluate the same pressure gradient term, an erroneous remainder can be produced, large compared to the change of geopotential along the constant pressure surface.

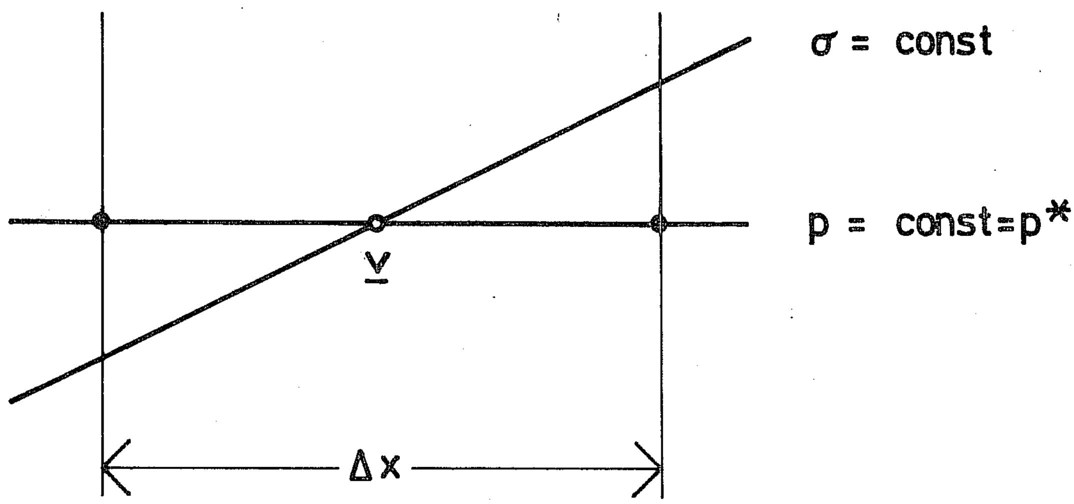

Fig. 1 Illustration of the hydrostatic consistency problem of the sigma coordinate schemes.

Another technique introduced by Janjić (1977), and recently proposed also by Arakawa and Suarez (1983), is directed at the accuracy of vertical differencing in the hydrostatic equation. Note that the geopotential of a pressure surface is given by

$$\phi = \phi_s - \int_{p_s}^{p} RT \; d \; \ln p \tag{1.7}$$

Thus, the geopotential of a constant pressure surface depends only on temperatures at and below the considered surface. If this property of the continuous equations is to be maintained by the finite-difference scheme, it will not be possible to use space-entered schemes of a high order of accuracy for the numerical integration of the hydrostatic equation. Indeed, with almost no exceptions, simple divided differences are used. In this situation, it was pointed out by Janjić that the actual accuracy of the vertical differencing and the resulting pressure gradient force can be increased if the variable used for the vertical differencing is judiciously chosen so as to minimize the error. In other words, (1.2) can be replaced by

$$\frac{\partial \phi}{\partial \zeta} = - \frac{RT}{p \; d\zeta/dp} , \tag{1.8}$$

where

$$\zeta = \zeta(p) \tag{1.9}$$

is a monotonic function of pressure, not necessarily equal to the vertical coordinate of the model. Instead, it is chosen so as to optimize the error properties of the scheme. This can be done with the idea of, once again, eliminating the error completely for a specific atmosphere. However, the technique can also be aimed at minimizing, in some average sense, the error of the difference approximation to $\partial\phi/\partial\zeta$ for a variety of the expected profiles.

An earlier suggestion of Phillips (1973) and Gary (1973) was to formulate the pressure gradient force in terms of deviations from a suitably chosen reference state. The effect of this technique is, in fact, equivalent to that of the Janjić and Arakawa-Suarez method. In recent experiments of Johnson and Uccellini (1983) these two methods have indeed given extremely similar results.

Most of these points we shall discuss in more detail in the continuation of this lecture.

2. A GENERAL FORM OF THE PRESSURE GRADIENT FORCE

We shall need an expression for the pressure gradient force enabling use of the separate hydrostatic equation coordinate (1.9), in addition to a general vertical coordinate

$$\eta = \eta(p, p_s, z) \tag{2.1}$$

η is also assumed to be a monotonic function of pressure (and/or geometric height, z); it can be a terrain-following (sigma), but also a more general type of coordinate. The choice of variables displayed here is motivated by an actual definition of η used for a specific pupose later on; note that a still wider selection of variables would not, in fact, change the considerations to follow.

To derive the general formula for the pressure gradient force in such an η-system, consider the situation schematically represented in Fig.2. Namely, using the notation introduced in the figure, we may write

$$-\frac{\phi_2-\phi_1}{\Delta s} = -\frac{\phi_3-\phi_1}{\Delta s} - \frac{\phi_2-\phi_3}{\Delta\zeta}\frac{\Delta\zeta}{\Delta s}\ .$$

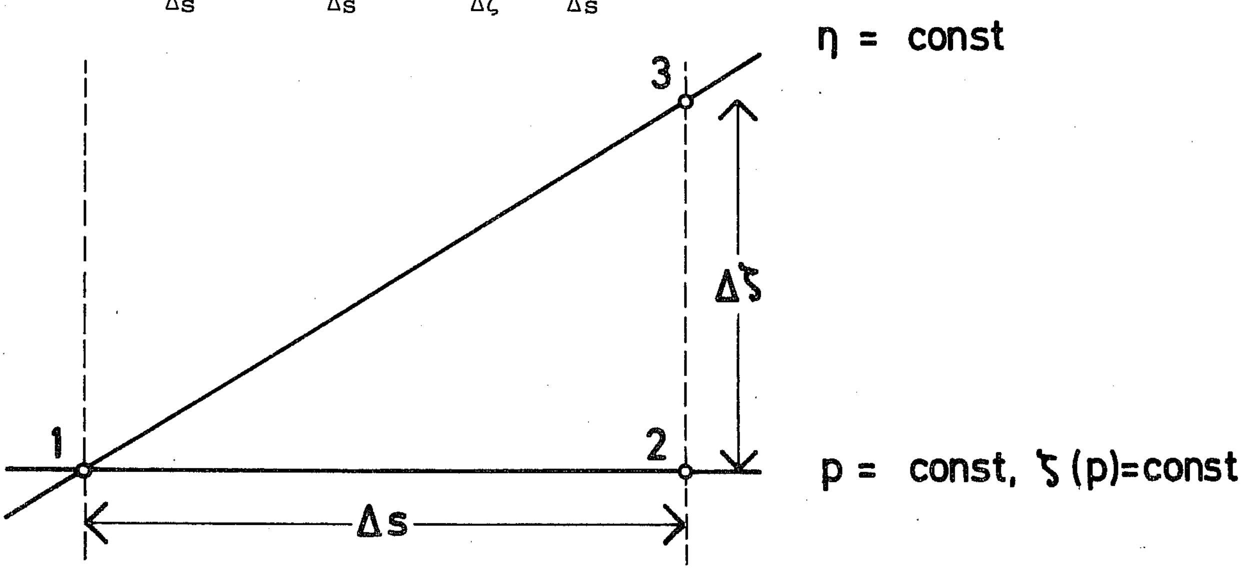

Fig. 2 Stencil and notation used to derive the general formula for the pressure gradient force in the eta (e.g. sigma) system.

If Δs is oriented in the direction of the largest variation of geopotential along the pressure surface, in the limit as Δs tends to zero this expression tends to

$$- \nabla_p \phi = - \nabla_\eta \phi + \frac{\partial \phi}{\partial \zeta} \nabla_\eta \zeta \qquad (2.2)$$

Here

$$\frac{\partial}{\partial \zeta} \equiv \frac{1}{\partial \zeta / \partial \eta} \frac{\partial}{\partial \eta} , \qquad (2.3)$$

that is, eta is the vertical coordinate actually used.

The formula (2.2) represents the general form of the pressure gradient force in the eta (e.g., sigma) system in the sense that any of the commonly used expressions can be derived from it by a particular choice of one or both of the functions ζ and η. For instance, if we choose

$$\zeta = \ln p \qquad (2.4)$$

formula (2.2) takes the form

$$- \nabla_p \phi = - \nabla_\eta \phi - RT \nabla_\eta \ln p \qquad (2.5)$$

(e.g., Simmons and Burridge, 1981). As an example where both of these functions are specified, consider the choice

$$\zeta = p; \quad \eta = \sigma = \frac{p - p_T}{\pi} , \ \pi \equiv p_s - p_T . \qquad (2.6)$$

It results in

$$- \nabla_p \phi = - \nabla_\sigma \phi + \frac{\sigma}{\pi} \frac{\partial \phi}{\partial \sigma} \nabla_\sigma \pi , \qquad (2.7)$$

which is another frequently used expression (e.g. Arakawa and Lamb, 1977).

One form of the pressure gradient force not resulting from (2.2) in a direct way is the "flux form" of Johnson (1980; Johnson and Uccellini, 1983)

$$\frac{1}{\pi} \left[- \frac{\partial}{\partial \sigma} (p \nabla_\sigma \phi) + \nabla_\sigma (p \frac{\partial \phi}{\partial \sigma}) \right] \qquad (2.8)$$

This form can however be obtained by a manipulation of the right hand side of (2.7). A straightforward discretization of the first term in the bracket of (2.8) again leads to a geopotential gradient term, same as that obtained by a discretization of (2.7).

3. DISCRETIZATION OF THE PRESSURE GRADIENT FORCE AND CALCULATION OF GEOPOTENTIAL BY THE HYDROSTATIC EQUATION

Following Janjić (1979), it will be convenient to consider the construction of the pressure gradient force scheme as consisting of three steps. In the first step the geopotential is calculated at constant η surfaces integrating the hydrostatic equation. As the second step, starting from known values at the neighbouring η surfaces, the geopotential is extrapolated or interpolated to a constant pressure surface, again via the hydrostatic equation. Finally, in the third step, the values of geopotential obtained in this way are used to calculate the finite-difference pressure gradient force approximation.

In order to apply this procedure, a specific distribution of variables over grid points in the vertical is needed. Almost all models carry temperatures (or potential temperatures) at the levels of horizontal velocity components, and the vertical velocities in between (the "Lorenz distribution" according to Arakawa). Among these models, differences exist in schemes used to calculate geopotential. Possibly all schemes can in this respect be classified as being either schemes which calculate geopotentials at the same levels as horizontal velocities, or schemes which calculate geopotentials at levels in between horizontal velocities. We shall call these schemes "non-staggered" (or "level"), and "staggered" (or "layer") schemes, respectively. These two possibilities are schematically represeneted in Fig.3. The symbol k appearing in the figure represents the vertical index, and $\delta_\zeta \phi$ is the finite-difference approximation to $\partial\phi/\partial\zeta$. It is defined by the particular form of the hydrostatic equation used.

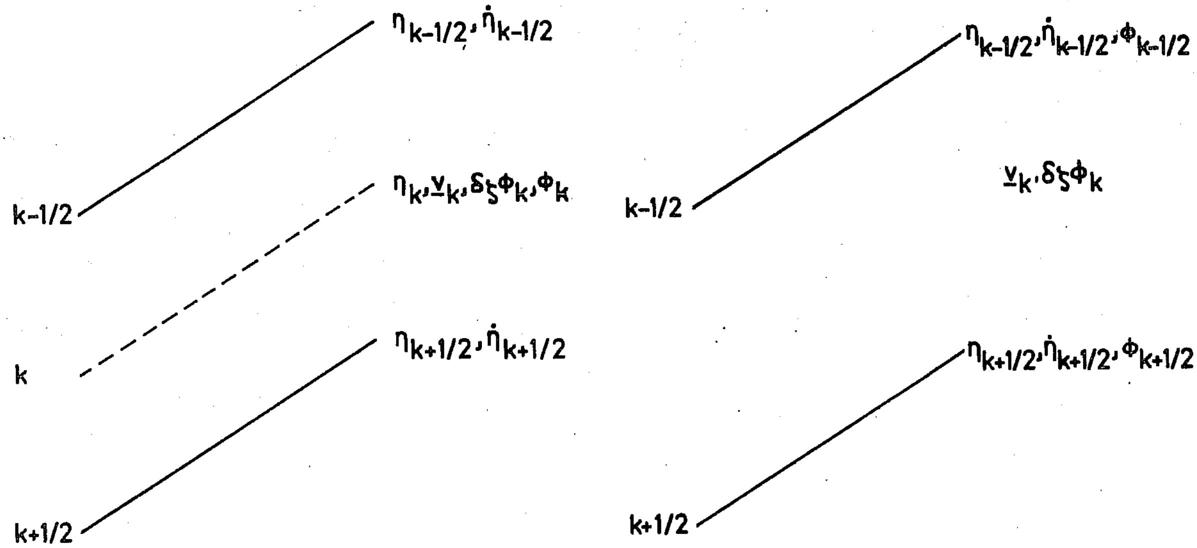

Fig. 3 "Non-staggered", or "level" (left panel) and "staggered", or "layer" (right panel) scheme for calculation of geopotential by the hydrostatic equation.

One should however note that a finite-difference hydrostatic equation for calculation of, for example, "full level" geopotentials, when it is supplied with a definition of "half level" (interface) geopotentials, can be rewritten so as to obtain a hydrostatic equation in terms of half level geopotentials, and a definition of full level geopotentials. The same kind of reformulation can be done starting with this latter set of equations, a hydrostatic equation for half level geopotentials and a definition of layer geopotentials. Thus, for a given scheme, it may not be obvious what should be considered as levels at which geopotentials are calculated by the finite-difference hydrostatic equation.

Given such a choice, by definition, we shall assume that the discrete values of $\phi(p)$ are actually defined at the points between which with the given finite difference hydrostatic equation we obtain the thicknesses of the layers, which, in the case of arbitrary spacing of vertical levels, do not depend on the temperatures outside of the layers, or depend on them to the minimum extent possible. Thus, as examples of non-staggered schemes the Arakawa

(1972), and the Corby et al. (1972) schemes can be given; staggered schemes are those of Janjić (1977), Burridge and Haseler (1977), and the Arakawa and Suarez (1983) "local hydrostatic equation" schemes.

Let us first consider the probably more frequently used, non-staggered scheme. For simplicity, we shall restrict ourselves to, say, the x component of the pressure gradient force. Let the x component of the pressure gradient force be calculated at the pressure level p*, and let the values of ζ and geopotential at this level be denoted by ζ^* and ϕ^* respectively. Furthermore, we shall assume that the first step has been completed and concentrate on the second and third step only. Using the notation introduced in Fig.4 we may write

$$\begin{aligned} \phi_1^* &= \phi_1^k + \delta_\zeta \phi_1^k (\zeta^* - \zeta_1^k) \ , \\ \phi_2^* &= \phi_2^k + \delta_\zeta \phi_2^k (\zeta^* - \zeta_2^k) \end{aligned} \tag{3.1}$$

where subscripts denote the grid points in the horizontal, and the superscript k indicates the η level at which the variables are defined.

We have now completed the second step of our procedure for calculating the pressure gradient force. The third step yields

$$- \frac{\phi_2^* - \phi_1^*}{\Delta x} = - \frac{\phi_2^k - \phi_1^k}{\Delta x} + \frac{1}{2} (\delta_\zeta \phi_1^k + \delta_\zeta \phi_2^k) \frac{\zeta_2^k - \zeta_1^k}{\Delta x}$$

$$- [\zeta^* - \frac{1}{2} (\zeta_1^k + \zeta_2^k)] \frac{\delta_\zeta \phi_2^k - \delta_\zeta \phi_1^k}{\Delta x} \ ,$$

or in a more compact notation

$$- \delta_x \phi^* = - \delta_x \phi^k + \overline{\delta_\zeta \phi}^x \delta_x \zeta^k - (\zeta^* - \overline{\zeta^k}^x) \delta_x (\delta_\zeta \phi^k). \tag{3.2}$$

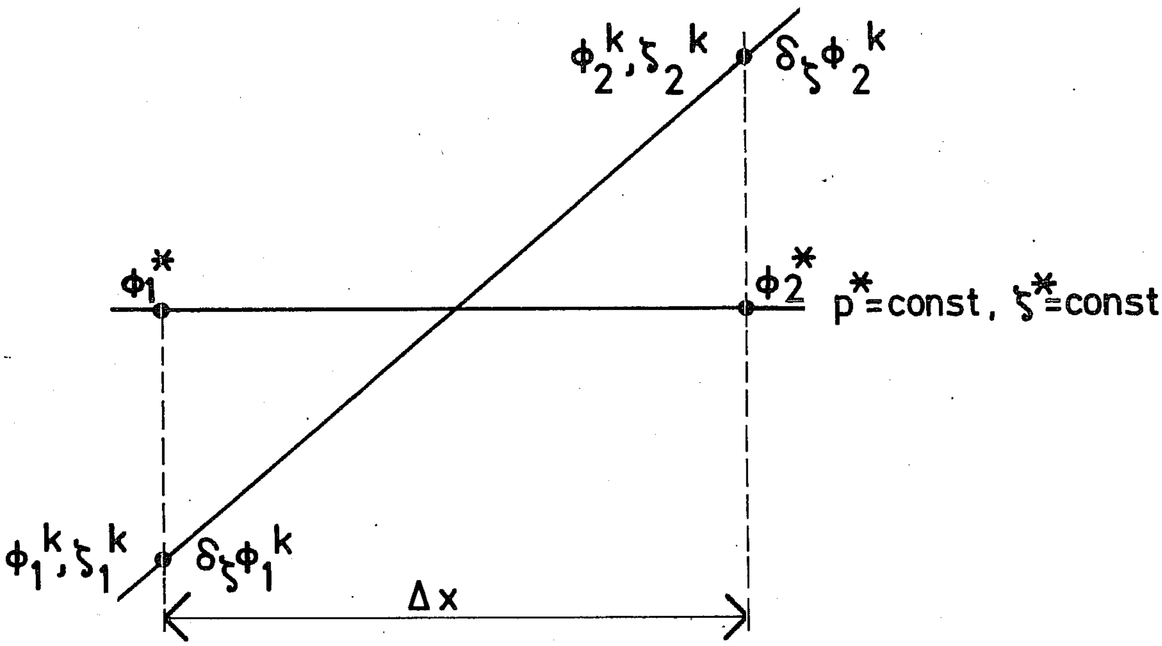

Fig. 4 Stencil and notation used to calculate pressure gradient force approximation in the case of the non-staggered scheme for calculation of geopotential by the hydrostatic equation.

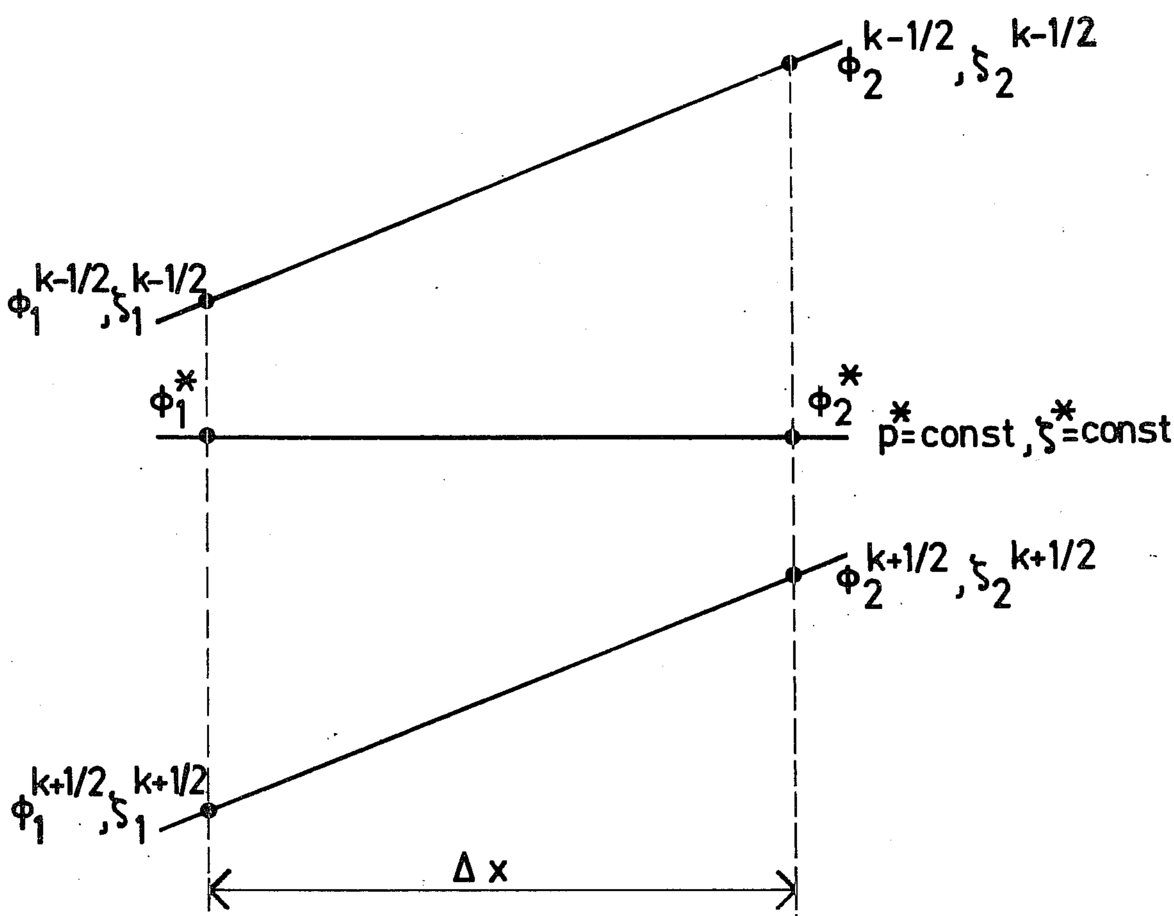

Fig. 5 Stencil and notation used to calculate pressure gradient force approximation in the case of the staggered scheme for calculation of geopotential by the hydrostatic equation.

If we define

$$\zeta^* = \overline{\zeta^k}^x \qquad (3.3)$$

we obtain the usual form of the pressure gradient force approximation with the non-staggered scheme

$$-\delta_x\phi^* = -\delta_x\phi^k + \overline{\delta_\zeta\phi}^x\delta_x\zeta^k \qquad (3.4)$$

As we can see from (3.3), unles ζ is a linear function of η, the approximation (3.4) is not defined at the level η_k as it is usually believed.

Let us now turn our attention to the staggered scheme. In the second step of our procedure for calculation of the pressure gradient force, we shall here use linear interpolation to obtain the geopotential at the pressure level p* corresponding to $\zeta^* = \zeta(p^*)$. Thus, with the notation introduced in Fig.5, we may write

$$\phi_1^* = \phi_1^{k-\frac{1}{2}} + \frac{\phi_1^{k+\frac{1}{2}} - \phi_1^{k-\frac{1}{2}}}{\zeta_1^{k+\frac{1}{2}} - \zeta_1^{k-\frac{1}{2}}}(\zeta^* - \zeta_1^{k-\frac{1}{2}}) ,$$
$$\phi_2^* = \phi_2^{k-\frac{1}{2}} + \frac{\phi_2^{k+\frac{1}{2}} - \phi_2^{k-\frac{1}{2}}}{\zeta_2^{k+\frac{1}{2}} - \zeta_2^{k-\frac{1}{2}}}(\zeta^* - \zeta_2^{k-\frac{1}{2}}) . \qquad (3.5)$$

Defining

$$\overline{A}^\eta \equiv \frac{1}{2}(A^{k-\frac{1}{2}} + A^{k+\frac{1}{2}}) , \qquad \delta_\zeta A \equiv \frac{A^{k+\frac{1}{2}} - A^{k-\frac{1}{2}}}{\zeta^{k+\frac{1}{2}} - \zeta^{k-\frac{1}{2}}} ,$$

these relations, after rearrangement, can be written as

$$\phi_1^* = \overline{\phi}_1^\eta + \delta_\zeta\phi_1(\zeta^* - \overline{\zeta}_1^\eta) ,$$

$$\phi_2^* = \overline{\phi}_2^\eta + \delta_\zeta\phi_2(\zeta^* - \overline{\zeta}_2^\eta) .$$

Proceeding with the third step, we find that

$$-\delta_x\phi^* = -\delta_x\overline{\phi}^\eta + \overline{\delta_\zeta\phi}^x\,\delta_x\overline{\zeta}^\eta - (\zeta^* - \overline{\overline{\zeta}^\eta}^x)\,\delta_x(\delta_\zeta\phi). \qquad (3.6)$$

If we define the level at which the pressure gradient force is calculated by

$$\zeta^* \equiv \overline{\overline{\zeta}^{\eta}}^{x} , \tag{3.7}$$

we finally obtain

$$- \delta_x \phi^* = - \delta_x \overline{\phi}^{\eta} + \overline{\delta_\zeta \phi}^{x} \, \delta_x \overline{\zeta}^{\eta} \tag{3.8}$$

Again we see that the relation (3.7), analogous to (3.3), was necessary in order to arrive at the result usually assumed.

In constructing the pressure gradient force schemes various authors, of course, typically did not follow the three step procedure outlined here. Usually, a difference analog to the hydrostatic equation would be chosen, and, subsequently, a difference approximation made to one of the differential expressions for the pressure gradient force. It is possible however to demonstrate that, in spite of the difference in approach, each of the commonly used schemes does in fact implicitly consist of the three step procedure described here. We shall have a look at some specific schemes, from that point of view, in later sections.

4. HYDROSTATIC CONSISTENCY

As we have seen, the first and the second step of our procedure for calculating pressure gradient force involve the integration of the hydrostatic equation in order to obtain the values of geopotential at constant η and constant pressure surfaces, respectively. In the first step, integration of the finite-difference hydrostatic equation will at each point of the horizontal grid define a certain vertical profile of geopotential, as a function of the integration variable ζ. Typically, as stated in the introductory section, this integration is done by approximating $\partial\phi/\partial\zeta$ in between discrete η_k levels by a constant, so that the resulting vertical profile of geopotential will be a piecewise linear function of ζ, as schematically represented in Fig.6.

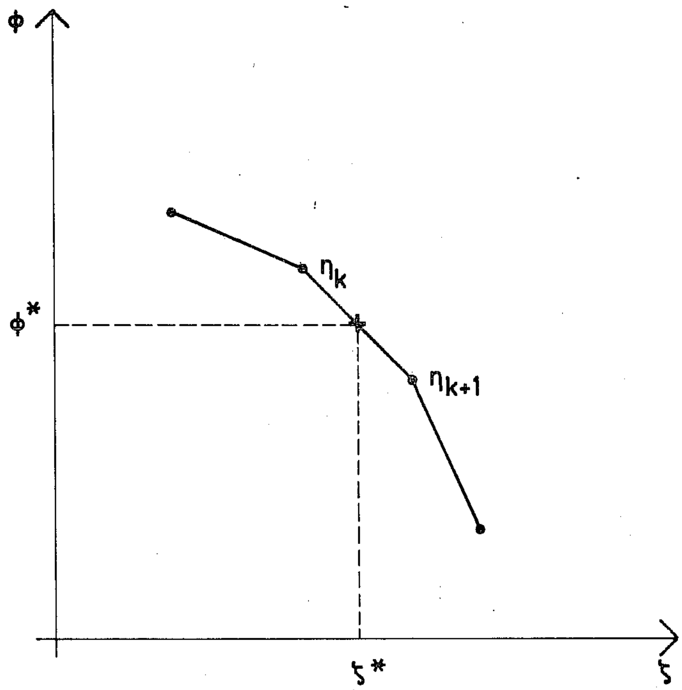

Fig. 6 Schematic representation of the vertical profile of geopotential defined by the finite-difference hydrostatic equation.

As proposed by Janjić, we can now define a scheme to be hydrostatically consistent if the procedure used in the second step yields a value of geopotential which lies on the geopotential profile defined by the finite-difference hydrostatic equation of the first step. For example, this could be the value represented by the plus sign in Fig.6. For hydrostatic consistency, in the second step, explicitly or implicitly, the same finite-difference hydrostatic equation has to be used, and the same grid point values, as in the first step.

Let us now, from the point of view of preceding considerations, have a look at some specific schemes for the hydrostatic equation and the pressure gradient force. We shall first consider possibly the simplest scheme with the non-staggered calculation of geopotential by the hydrostatic equation, that of Corby et al. (1972; also Gilchrist, 1975; and Corby et al. (1977). They have used the original definition of the sigma coordinate, (1.4), and a non-staggered arrangement of horizontal velocities, temperatures and geopotentials both in horizontal and in the vertical direction (Fig.7). The geopotential of the lowest level (k=K) they have defined by

$$\phi_K = \phi_s + RT_k \ln(1/\sigma_K) \tag{4.1}$$

and the difference analog of the hydrostatic equation above that level by

$$\Delta_\sigma \phi = - R\overline{T}^\sigma \Delta_\sigma \ln\sigma \ . \tag{4.2}$$

Thus,

$$\phi_k = \phi_s + \sum_{n=k}^{K-1} \frac{1}{2} R(T_n + T_{n+1}) \ln \frac{\sigma_{n+1}}{\sigma_n} + RT_K \ln \frac{1}{\sigma_K} \tag{4.3}$$

defines the vertical profile of geopotential prescribed by the finite-difference hydrostatic equation, that is, by the first step of our procedure.

To choose the analog of the pressure gradient force, Corby et al. adopt

$$\delta_x \overline{\phi}_k^x \tag{4.4}$$

as the difference approximation to $(\partial\phi/\partial x)_\sigma$, and seek to find a consistent approximation to the pressure gradient term so that combined the two terms give a zero pressure gradient in case of the resting atmosphere defined by

$$T(p) = A\ln p + B \tag{4.5}$$

The profile (4.5) they have considered a reasonable average profile for the troposphere. To this end, they substitute (4.5) into the hydrostatic equation

$$d\phi = - RT \ d \ln p \ , \tag{4.6}$$

and, integrating, obtain

$$\phi(p) = \phi_o + \frac{1}{2} RA \ (\ln^2 p_o - \ln^2 p) + RB \ (\ln p_o - \ln p) \ . \tag{4.7}$$

Thus, ϕ_s and p_s are seen to be related by

$$\phi_s = \phi_o + \frac{1}{2} RA \ (\ln^2 p_o - \ln^2 p_s) + RB \ (\ln p_o - \ln p_s) \tag{4.8}$$

Corby et al. now evaluate (4.4) by inserting geopotentials given by (4.3) and (4.8), and sigma level temperatures obtained from (4.5), that is

$$T_n = A\ln(\sigma_n p_s) + B \tag{4.9}$$

After some algebra, they arrive at

$$\delta_x \overline{\phi}_k^x = - \overline{R\overline{T}_k^x \ \delta_x \ \ln p_s}^x \tag{4.10}$$

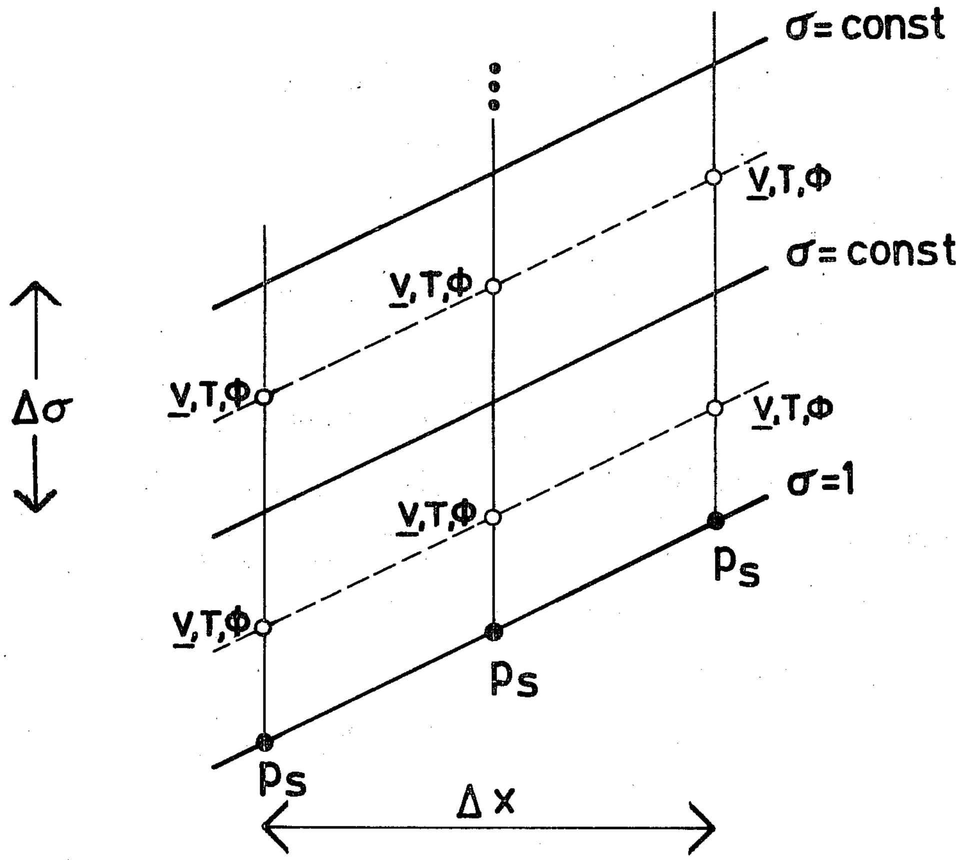

Fig. 7 The vertical grid of the Corby et al. (1972) scheme, with a non-staggered arrangement of the time dependent variables p_s, $\underline{V}$ and T, and locations where ϕ is defined by the finite-difference scheme.

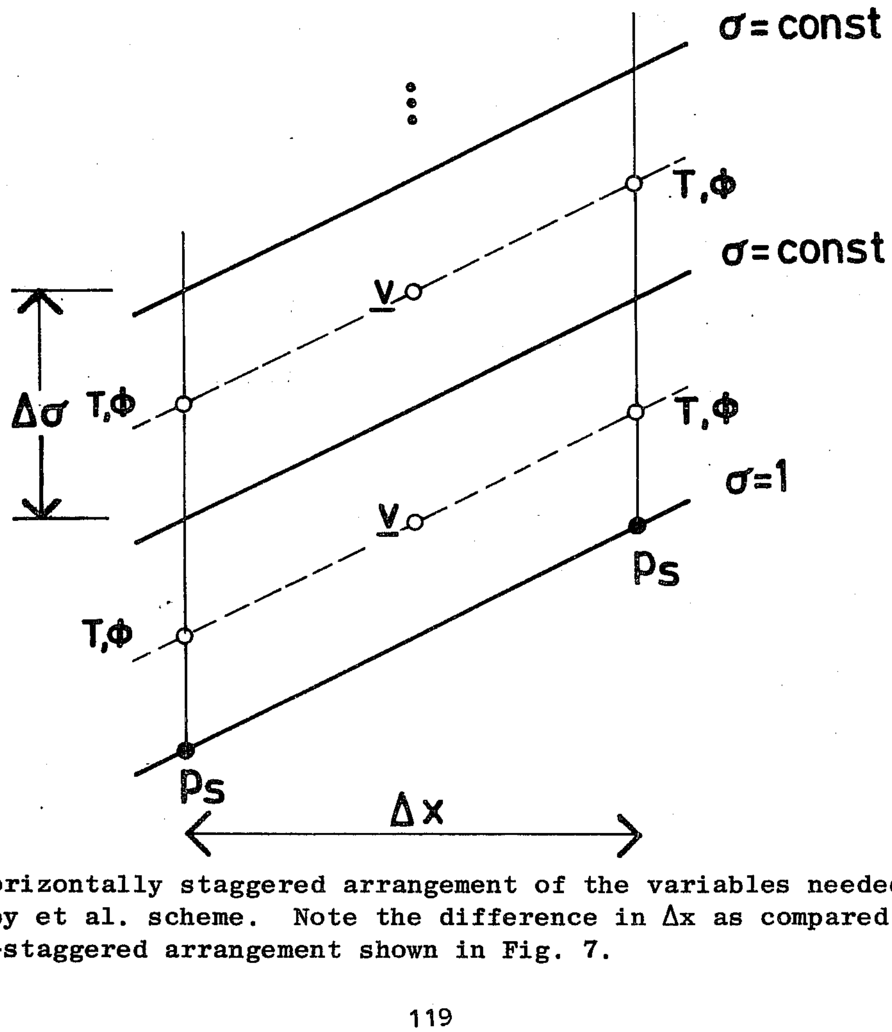

Fig. 8 A horizontally staggered arrangement of the variables needed by the Corby et al. scheme. Note the difference in Δx as compared to the non-staggered arrangement shown in Fig. 7.

Accordingly, to eliminate completely the error in the case of the atmosphere which is being considered, Corby et al. choose

$$- \overline{R\overline{T}_k^x \, \delta_x \ln p_s}^x \tag{4.11}$$

as the difference analog of the x component of the pressure gradient term.

For further simplification, let us consider the version of the Corby et al. scheme obtained using a horizontally staggered grid, with velocity and temperature and geopotential defined at alternate grid points (Fig.8). Then, following the same procedure, one arrives at the analog

$$- \delta_x \phi_k - R\overline{T}_k^x \, \delta_x \ln p_s \tag{4.12}$$

For a given value of Δx, (4.12) has a smaller truncation error than the original scheme.

Now to obtain the scheme (4.12) using our three step procedure, we shall as the second step need the extrapolation scheme

$$\begin{aligned} \phi_1^* &= \phi_{1,k} - RT_{1,k} (\ln p^* - \ln p_{1,k}) , \\ \phi_2^* &= \phi_{2,k} - RT_{2,k} (\ln p^* - \ln p_{2,k}) , \end{aligned} \tag{4.13}$$

As could be expected in view of the second term of (4.12), only two grid point values of temperature appear in (4.13). As the third step, the expression

$$- \frac{\phi_2^* - \phi_1^*}{\Delta x} \tag{4.14}$$

is evaluated. Inserting (4.13), assuming that

$$\ln p^* \equiv \frac{1}{2} (\ln p_{1,k} + \ln p_{2,k}) \tag{4.15}$$

and, finally, taking into account that

$$\delta_x \ln p = \delta_x \ln p_s$$

one indeed arrives at (4.12).

The values ϕ^* used in (4.13) are, however, not on the geopotential profile (4.3) defined by the finite-difference hydrostatic equation, (4.2), and, thus, the scheme is hydrostatically inconsistent. The finite-difference hydrostatic equation implied by (4.13), involving single sigma level temperatures only, is different from the hydrostatic equation (4.2), involving two point vertical averages of temperature. In addition, in case of steep sigma surfaces and/or thin sigma layers, the pressure surface $p^* = \text{const}$ can intersect additional sigma levels in between the grid points 1 and 2 used to form the analog (4.14).

Thus, rather than the values (4.13), a method consistent with the geopotential profile defined by (4.3) would have to use the values of geopotential located on that profile. Denoting these values by ϕ^c, and assuming that the pressure p^* is at grid points 1 and 2 located in between the levels k-r and k-r-1, and k+r and k+r+1, respectively, we have

$$\phi_1^c = \phi_{1,k-r} - \frac{1}{2} R\,(T_{1,k-r} + T_{1,k-r-1})(\ln p^* - \ln p_{1,k-r}),$$

$$\phi_2^c = \phi_{2,k+r} - \frac{1}{2} R\,(T_{2,k+r} + T_{2,k+r+1})(\ln p^* - \ln p_{2,k+r}) \qquad (4.16)$$

with the values $\phi_{1,k-r}$ and $\phi_{2,k+r}$ here given by (4.3). In this way, with the extrapolation prescribed by (4.16) performed along the piecewise linear profile of geopotential defined by the hydrostatic equation scheme, evaluation of

$$-\frac{\phi_2^c - \phi_1^c}{\Delta x} \qquad (4.17)$$

gives the pressure gradient force with the vertical discretization as accurate as it can be achieved within the accuracy limits of the hydrostatic equation scheme.

In the simple case r=0, when the slope of the sigma surface is not too great and/or the considered sigma layers are not too thin, so that the surface p* intersects no additional sigma levels in between the grid points 1 and 2, (4.17) gives

$$- \delta_x \phi_k - \frac{1}{4} R \,(T_{j-1,k-1} + T_{j-1,k} + T_{j+1,k} + T_{j+1,k+1}) \delta_x \ln p_s . \quad (4.18)$$

Here, of course, again (4.15) had to be assumed, and j represents the grid point index along the x direction.

The difference between the Corby et al. scheme (4.12) and the hydrostatically consistent analog (4.18)

$$\frac{1}{4} R \,(T_{j+1,k+1} + T_{j+1,k} + T_{j-1,k} + T_{j-1,k-1}) \delta_x \ln p_s \quad (4.19)$$

can be considered to represent the error of (4.12) due to its hydrostatic inconsistency. It can be written as

$$\frac{1}{4} R \left[\left(\frac{\Delta_\sigma T}{\overline{\Delta\sigma}^\sigma} \overline{\Delta\sigma}^\sigma \right)_{j+1,k+\frac{1}{2}} - \left(\frac{\Delta_\sigma T}{\overline{\Delta\sigma}^\sigma} \overline{\Delta\sigma}^\sigma \right)_{j-1,k-\frac{1}{2}} \right] \delta_x \ln p_s \quad (4.20)$$

and is thus, formally, a small quantity. Nevertheless, the presence of (4.19) seems to be a rather undesirable feature of the scheme. Grid point values of temperature are known to exhibit large grid point to grid point variations due to the effects of the so-called physical processes in atmospheric models, and the factor $\delta_x \ln p_s$ increases in magnitude with the slope of the model mountains. Scheme (4.18), free of this error, appears from that point of view certainly much more appealing. It has, however, the problem of being asymmetric, and thus possibly time consuming. In addition, it is not applicable at the uppermost as well as at the lowermost level of the model.

The staggered scheme for calculation of geopotential permits a straightforward construction of hydrostatically consistent pressure gradient force schemes without these disadvantages. As an example, consider the scheme of Burridge and Haseler (1977). They use the same arrangement as that in Fig.8, except that geopotentials are carried at the interfaces of sigma layers. As the hydrostatic equation they have simply

$$\Delta\phi_k = -RT_k \Delta\ln\sigma_k \tag{4.21}$$

for all the sigma layers. Here

$$\Delta A_k \equiv A_{k+\frac{1}{2}} - A_{k-\frac{1}{2}}$$

denotes the vertical difference across the layer. The pressure gradient force retains the form (4.12), with the layer values of geopotential, needed by (4.12), prescribed as

$$\phi_k \equiv \frac{1}{2}\left(\phi_{k-\frac{1}{2}} + \phi_{k+\frac{1}{2}}\right) \tag{4.22}$$

To construct this scheme using the described three step procedure, again equations (4.13) are needed as the second step. However, now the finite-difference hydrostatic equation implied by these equations is the same as the hydrostatic equation used to calculate geopotentials, (4.21). Thus, in this sense, the scheme is hydrostatically consistent.

The staggered schemes of Janjić (1977), and Arakawa and Suarez (1983), in the same way as the Burridge and Haseler scheme, achieve hydrostatic consistency through hydrostatic equations of the (4.21) type, in which a single temperature (or potential temperature) governs the change of geopotential across a layer. As already mentioned, they have additional features which shall be discussed later on to some extent.

However, the second step equations of the type (4.13) can of course maintain hydrostatic consistency only as long as extrapolation is performed along linear segments of the profile of geopotential. Thus, as pointed out by Janjić (1977), hydrostatically consistent schemes of this type lose consistency in a situation as shown in Fig.9, when points at which the pressure surface p* intersects the neighbouring grid verticals are outside of the considered layer. These points are denoted by A and B in the figure.

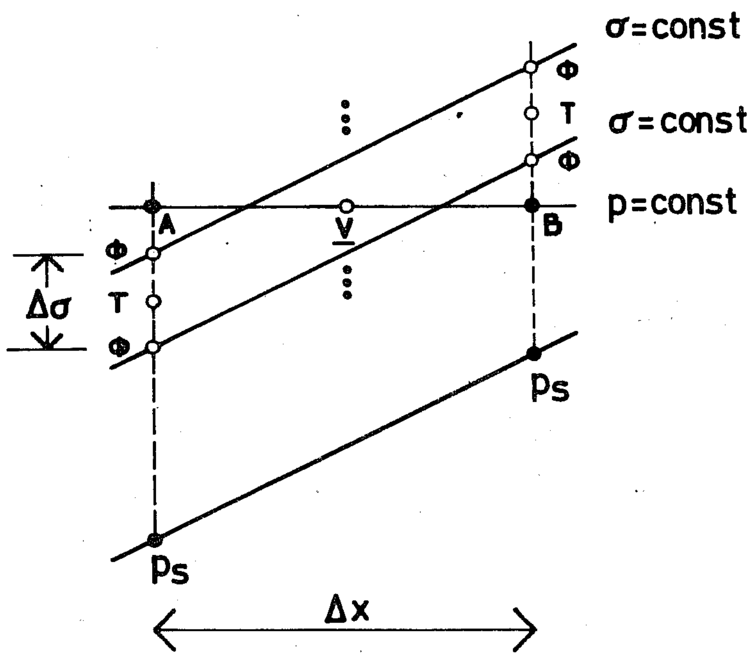

Fig. 9 An example of hydrostatically inconsistent evaluation of the pressure gradient force by schemes such as the Burridge and Haseler (1977), (4.21) and (4.12), scheme (Janjić, 1977).

As seen from the figure, for the hydrostatic consistency of this group of schemes one should require that

$$\left|\delta_x \phi\right|_\sigma \Delta x \leqslant \left|\delta_\sigma \phi\right| \Delta\sigma \qquad (4.23)$$

Thus, increasing the steepness of model mountains, and increasing the vertical resolution, may lead to a violation of the hydrostatic consistency of the scheme.

5. CONVERGENCE

The property of hydrostatic inconsistency appears to be under certain conditions related to the lack of convergence of the scheme, (Mesinger, 1982). An obvious example is the Arakawa (1972) scheme, in which the geopotential of the lowest level, and consequently, also those of all levels above the lowest level, depend on the temperature of all model levels. Since as pointed out in the introductory section, the true value of the pressure gradient force does not depend on temperatures above the considered pressure surface, in such a situation the difference between the true value of the pressure gradient force and its difference approximation can be arbitrarily great. This cannot be removed by the reduction of the grid intervals, and, hence, the scheme is not convergent (Arakawa and Suarez, 1983).

A different kind of problem exists with the Corby et al. scheme. It is illustrated with the help of Fig.10. Note that the "domain of dependence" of the true value of the pressure gradient force at the $\underline{V}$ point in that figure is the region below the line p=const shown in the figure. On the other hand, the finite-difference value calculated at the same point using the Corby et al. scheme depends on values at grid points on and below the dashed line connecting the two T,ϕ grid points. In this way, a section exists of the domain of dependence of the true value of the pressure gradient force which is not included in the domain of dependence of its finite-difference value. If the temperature gradient is permitted to be a discontinuous function of space coordinates, the error of the scheme can again be arbitrarily great.

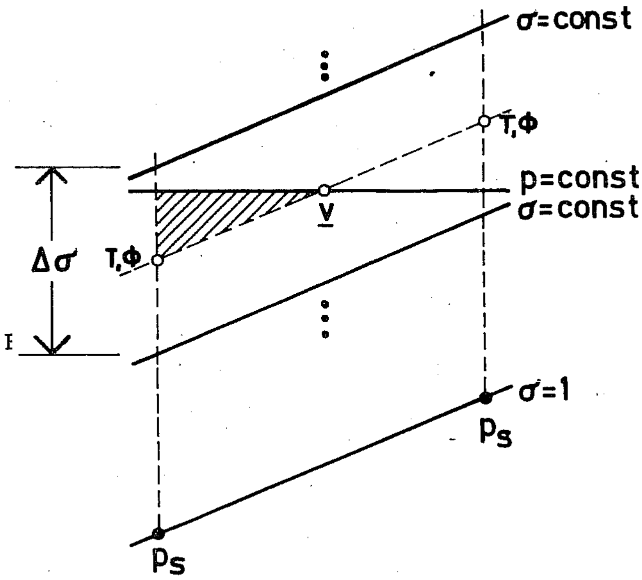

Fig. 10 Illustration of the convergence problem of Corby et al. scheme. The shaded area represents the section of the domain of dependence of the true value of the pressure gradient force not included in the domain of dependence of its finite difference value.

With schemes using the staggered calculation of geopotential, a similar situation occurs when the consistency condition (4.23) is violated. Then again a section exists of the domain of dependence of the true value of the pressure gradient force not included in the domain of dependence of its finite-difference value. In Fig.9, this is the section bounded by the vertical line at point A, the isobar and the uppermost constant sigma line shown in the figure.

6. THE ERROR PROBLEM: A NUMERICAL EXAMPLE

A numerical example calculated by Mesinger (1982) will be included here as an illustration of possible quantitative implications of some of the preceding considerations. Following a number of previous authors, an atmosphere in hydrostatic equilibrium is considered, so that the error is entirely due to the pressure gradient force approximation. Errors are calculated for the Corby et al. and for the Burridge and Haseler schemes, taken as possibly the simplest examples of hydrostatically inconsistent and consistent schemes, respectively.

Before describing the example, one more remark regarding the Corby et al. scheme should be made. Recall that the scheme was constructed to achieve exact cancellation of the two terms when temperature is a linear function of lnp, (4.5). The procedure used included insertion of sigma level temperatures, (4.9). An objection can however be raised regarding this step. Namely, schemes used to initialize finite-difference calculations are typically not based on an analysis of temperature; rather, they are based on the analysis of geopotential. Note that values of geopotential represent information on the vertically integrated temperature. Thus, observed values of geopotential, defining the true value of the pressure gradient force except for observational errors, cannot be recovered from a limited number of the local observations of temperature. Therefore, by analyzing temperature to initialize a primitive equation model, needlessly pressure gradient force errors are introduced in addition to those due to the observational errors and to the finite-difference scheme.

Accordingly, instead of using (4.9), it would have been more appropriate to use (4.7) to define the sigma level geopotentials, and then calculate temperature from the difference equations (4.1) and (4.2). Temperatures obtained in this way would differ from those defined by (4.9), and, therefore, the two terms of (4.12) would not cancel. Similar consideration is applicable to error estimates of a number of other authors (e.g., Nakamura, 1978; Simmons and Burridge, 1981).

To have the numerical example include an estimate of the resulting error, Mesinger has calculated errors for the reference atmosphere of Corby et al., in which temperature is a linear function of lnp. It seemed of interest to have a look also at errors in a case of a more irregular variation of temperature; for example, error calulations of Janjic (1977) showed maximum errors at tropopause levels. Therefore, a temperature variation including an inversion was also considered. Recalling a situation with cold air impinging at one side of a mountain barrier, an "inversion case" was defined, with an inversion below a presumed mountain height. In this inversion case, except for the inversion level, temperature was still assumed to be linear in lnp.

Following the error calculations of Phillips (1974) and Janjic, two neighbouring surface pressure points were considered located along the direction of the x axis at pressures of 1000 mb and 800 mb, respectively. The temperature at the 800 mb level was taken to be 0°C, and those at 1000 mb 10°C ("no inversion case") and -10°C ("inversion case"). Temperature above 800 mb in the "inversion case" was taken to be the same as in the "no inversion case". These two temperature profiles are shown in the left panel of Fig.11. Errors were calculated for the velocity point located in between the two surface pressure points, at (or, approximately at, the level σ=0.9, as sketched in the right panel of the figure. Calculations were performed for $\Delta\sigma$=1/5 (below the σ=0.8 interface), $\Delta\sigma$=1/(3x5), $\Delta\sigma$=1/(5x5), etc., and in the limit as $\Delta\sigma \to 0$. These values are chosen to keep the velocity point at the same altitude, at $\sigma \cong 0.9$.

The errors obtained for the two schemes and the two temperature profiles are displayed in Table 1. The values shown are in units of geopotential. To obtain the values of the pressure gradient force they should be divided by the horizontal grid interval.

Table 1 Errors of the pressure gradient force analogs obtained using the Corby et al. and the Burridge and Haseler scheme, for the "no inversion case" and the "inversion case"; see text for details. Values are given in increments of geopotential (m^2s^{-2}), between two neighbouring grid points, along the direction of the increasing terrain elevations.*

$\Delta\sigma$=	1/5	1/15	1/25	...	$\lim_{\Delta\sigma\to 0}$
Corby et al. scheme "no inversion case"	151.2	-48.7	29.0	...	0
Corby et al. scheme "inversion case"	-159.6	-159.6	-159.6	...	-159.6
Burridge and Haseler scheme "no inversion case"	0	0	0	...	0
Burridge and Haseler scheme "inversion case"	0	-142.1	-153.3	...	-159.6

*Note that some of the numbers in the last two lines are slightly different from those published in the referred paper; this is a result of the removal of an error that Mesinger has found in his program for calculation of the Burridge and Haseler scheme values. The numbers published previously actually referred to a scheme which within the geopotential gradient term used geopotentials of the σ=0.9 surface, rather than values defined by (4.22). Corrected values are given in the present table.

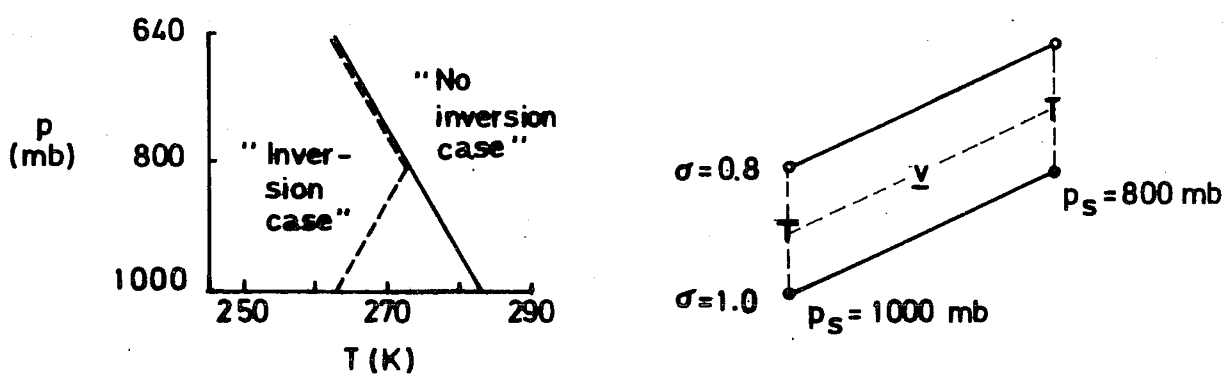

Fig. 11 The temperature profiles used to calculate the errors of the Corby et al. and of the Burridge and Haseler pressure gradient force scheme (left panel), and the location of the grid point at which the errors were calculated (right panel), Mesinger (1982).

Quite an appreciable error is found to be associated with the Corby et al. scheme in the "no inversion case", when, supposedly, the scheme should have had no error. The error, of course, results from the procedure used here to prescribe geopotentials, rather than temperatures. If the horizontal grid interval is taken to be 150 km the obtained error of about 150 m^2s^{-2}, expressed in terms of geostrophic wind, corresponds to an error of the order of 10 ms^{-1}. Since with the temperature profile of the "no inversion case" the error shown for the Corby et al. scheme is entirely due to the discretization of the hydrostatic equation, the error vanishes in the limit as the thicknesses of sigma layers tend to zero.

With modest vertical resolution ($\Delta\sigma$ = 1/5), the Corby et al. scheme is seen to give an error of about the same magnitude in the "inversion case". However, with the temperature not being a single linear function of lnp, the error now does not vanish as the thicknesses of sigma layers tend to zero. In fact, it can be demonstrated that in this particular example the vertical resolution has no effect on the error. Thus, rather large errors are shown to be possible with the Corby et al. scheme irrespective of the vertical discretization problem.

With the hydrostatically consistent Burridge and Haseler scheme there is no error in the "no inversion case". Having no error in the limit as $\Delta\sigma \rightarrow 0$ should come as no surprise: recall that the two schemes differ only in the discretization of the hydrostatic euation, and this difference vanishes as the thickness of sigma layers tends to zero. The absence of an error in case $\Delta\sigma$=1/5 can be understood in view of the hydrostatic consistency of the scheme, and the special configuration of the sigma layer in this case (the right hand panel of Fig.11), such that the pressure surface p* passes through geopotential grid points at interfaces of the layer. With the geopotential interpolated in a consistent way, along its piecewise linear profile defined by the finite-difference hydrostatic equation, exact interface values of geopotential are recovered. The reason for the absence of errors at higher (but still not infinite) vertical resolution, when the consistencey condition (4.23) is violated, is at this point not obvious. We shall return to an analysis of this case in the following section.

In the "inversion case", there is of course again no error for $\Delta\sigma=1/5$; note that the hydrostatic consistency argument just given does not involve the prescribed temperature profile. However, as the vertical resolution is increased the consistency condition is violated, and an appreciable error is once more obtained. The asymptotic error is entirely due to hydrostatic inconsistency: correct values of geopotential of the sigma surface, obtained by a vertical integration of the hydrostatic equation, are used to calculate the increment of geopotential within the first term of the pressure gradient force; while a local (incorrect) calculation of the sigma-to-pressure increments of geopotential is performed within the second term.

One should note that this rather large asymptotic error is a result of only the upper half of the inversion shown in Fig.11. In the considered limit the two schemes are not affected by the actual temperature profile within the lower half of the assumed inversion layer; a different profile in this lower half of that layer, provided the geopotential of the 900 mb remained the same, would have given the same asymptotic errors.

The asymptotic error of this example illustrates the nature of the pressure gradient force problem: it was not possible to remove this error by an increase in the accuracy of vertical differencing. In fact, it was demonstrated that an increase in the accuracy of vertical differencing can be associated with an increase in the error. The errors considered are related to the chosen horizontal grid length, but only through the difference in surface pressure between neighbouring grid points. Therefore, these errors cannot be removed by an increase in formal accuracy of the horizontal differencing. Thus, it is seen that the difficulty is not due to the truncation error of the approximations to the space derivatives involved, as it is frequently believed. Considered errors result from the two point temperature average used within the second term of the considered schemes, and also cannot be removed by an increase in formal accuracy of the temperature analog used within that term.

7. REDUCTION OF THE ERROR

On the basis of arguments presented so far we believe the use of hydrostatically consistent schemes certainly appears advisable. However, this generally still does not accomplish elimination of the error, even in the simple case of a resting atmosphere. Let us, therefore, proceed with the analysis of the error.

Consider the reason for the error in case of a hydrostatically consistent scheme and an atmosphere at rest. Recall the argument explaining the absence of the error of the Burridge and Haseler scheme in case $\Delta\sigma=1/5$ of the preceding example: a consistent interpolation of geopotential was performed up to the ends of two linear segments of the defined geopotential profile, so that exact geopotential values were recovered. In the more general case, such as illustrated in Fig.5, even though it is hydrostatically consistent interpolation will have an error, because the actual profile of geopotential is not the same as that defined by the finite-difference hydrostatic equation. Thus, the values ϕ^* will generally have errors. Furthermore, errors of

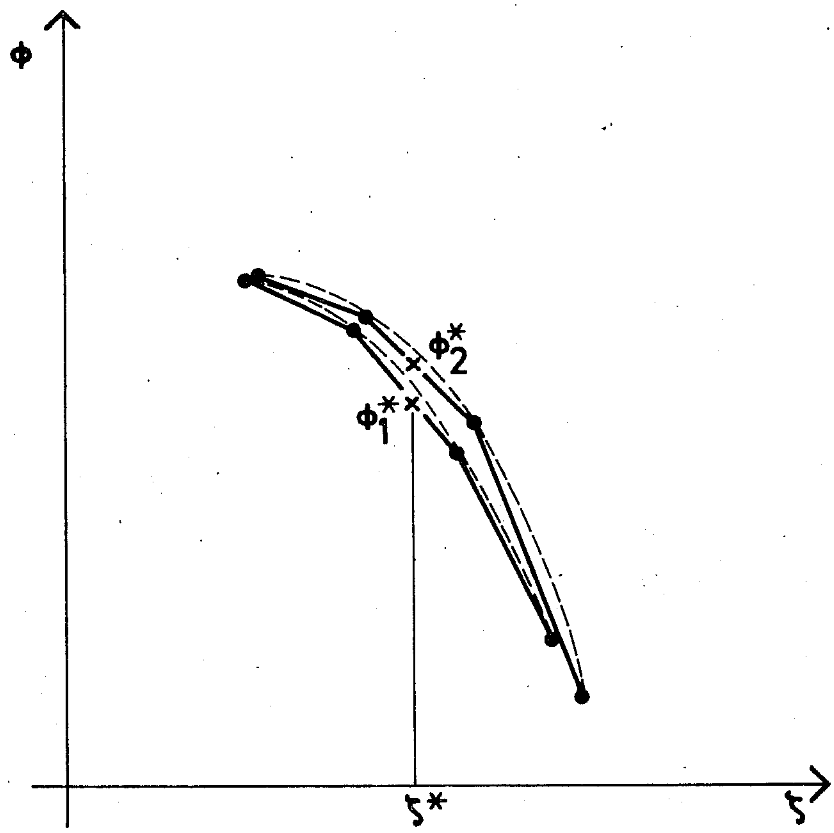

Fig. 12 A schematic representation of the geopotential profiles at two adjacent grid points in the sigma system (Janjić, 1979).

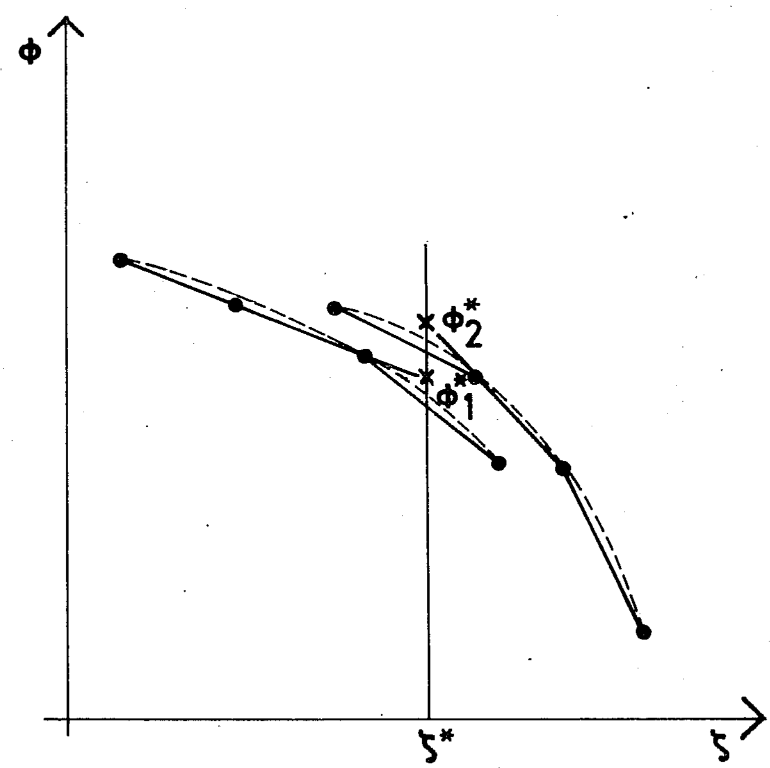

Fig. 13 Same as in Fig. 12 but for the case of hydrostatic inconsistency.

ϕ_2^* and ϕ_1^* will generally be different, so that

$$-\frac{\phi_2^* - \phi_1^*}{\Delta x}$$

will have an error.

In most cases, the geopotential profile defined by the finite-difference hydrostatic equation should be considered to be a piecewise linear function such as e.g. shown in Fig.6. The errors of ϕ^* would then be due to the deviation of the actual geopotential profile from that piecewise linear one. This situation is visualized in Fig.12. The dashed lines represent the "true" profiles at two adjacent points of the horizontal grid. The heavy dots correspond to the values of geopotential at the interfaces of the sigma layers while the two piece-wise linear curves represent the geopotential profiles obtained by integration of the finite-difference hydrostatic equation. As before, the values of ϕ_1^* and ϕ_2^*, denoted by crosses, are used to calculate the pressure gradient force at the pressure level defined by ζ^*.

So far we have discussed the pressure gradientg force error in the case of hydrostatically consistent calculations. The situation corresponding to the hydrostatic inconsistency is schematically represented in Fig. 13. The notation is the same as in Fig.12.

This time the error results from the linear extrapolation of geopotential beyond the range of validity of the finite difference approximation to $\frac{\partial\phi}{\partial\zeta}$. Apparently, in this case we should expect larger errors.

A similar analysis can be performed to show that analogous problems are encountered in the calculations using non-staggered schemes for calculation of geopotential.

An obvious question at this point is - can one eliminate this error? It has been pointed out by Janjić (1979) that linear interpolation/extrapolation as illustrated by preceding figures will be error-free if ζ is chosen so as to have ϕ a linear function of ζ. In principle, this of course can be done for any given geopotential profile. However, note that in the "no inversion case" of the previous section for the Burridge and Haseler scheme, for which $\zeta=\ln p$, no errors were obtained even though geopotential was a quadratic and not a linear function of ζ. One may wonder whether some additional guidance can be obtained from an understanding of the reason for the absence of the error in this case.

Consider, therefore, a general quadratic function

$$\phi = \alpha + \beta\zeta + \gamma\zeta^2 \tag{7.1}$$

where α, β and γ are constants, and $\gamma\neq 0$. Note that, without loss of generality, this can be written as

$$\phi = C + \frac{1}{2B}(A + B\zeta)^2 , \tag{7.2}$$

with A, B, C being a new set of constants. We are interested in errors of the scheme (3.8)

$$- \delta_x \overline{\phi}^{\eta} + \overline{\delta_\zeta \phi}^{x} \delta_x \overline{\zeta}^{\eta} \tag{7.3}$$

with geopotentials at interfaces being prescribed by (7.2).

When we insert (7.2) into the first term of (7.3) we readily obtain

$$- \delta_x \overline{\phi}^{\eta} = - \overline{\overline{(A + B\zeta)}^{x} \delta_x \zeta}^{\eta} \, . \tag{7.4}$$

Similarly, evaluation of the difference in geopotential across the layer yields

$$\Delta\phi_{\eta} = \overline{(A + B\zeta)}^{\eta} \Delta\zeta_{\eta} \, ,$$

so that we have at the same time

$$\overline{\delta_{\zeta}\phi}^{x} \delta_x \overline{\zeta}^{\eta} = \overline{\overline{(A + B\zeta)}^{\eta}}^{x} \delta_x \overline{\zeta}^{\eta} \, . \tag{7.5}$$

Inspection of the right hand sides of (7.4) and (7.5) shows that they will cancel when $\delta_x \zeta$ is not dependent on η; thus, in that case the scheme (7.3) will have no error. The Burridge and Haseler scheme is a special case of this situation: it is obtained from (7.3) inserting $\eta = p/p_s$ and $\zeta = \ln p$, which satisfies the requirement for $\delta_x \zeta = \delta_x \ln p_s$ to be independent of η. This, therefore, explains the absence of the error of this scheme in the "no inversion case" of the preceding section. It is thus seen that with geopotential considered specified it is the Burridge and Haseler scheme that has no error when temperature is a linear function of lnp; while it was the Corby et al. scheme that in that case had no error with temperature considered specified.

Another special case for which the right hand sides of (7.4) and (7.5) cancel is the case B=0. This case was not covered by the analysis just given, which assumed $\gamma = \frac{1}{2}B \neq 0$. However, note that for $\gamma=0$ ϕ is a linear function of ζ; thus, (7.3) will indeed have no error in this case as well.

It is not obvious that much guidance has been obtained from this consideration. The feature of a scheme to have no error for a general quadratic function of ζ, as compared to a linear function only, certainly seems attractive. However, there is no guarantee that this feature will be associated with acceptable errors in all, or most, cases of interest. For example, the Burridge and Haseler scheme has indeed had quite large errors in the "inversion case" of the preceding section. Speaking more generally, it is not clear in what way or even if making the error zero for a specific stratification reduces the error in general. In addition, reduction of the error may not be the only point to consider when deciding on the choice of ζ. For example, various choices could substantially differ from the point of view of computational economy.

An attempt to deal with both of these difficulties has been made by Janjic (1977). He has considered a family of schemes obtained from (7.3) by inserting

$$\eta = \frac{p - p_T}{p_S - p_T}, \quad \zeta = (\ln p)^{1+m}, \tag{7.6}$$

and has then sought to optimize the choice of the parameter m. To this end, for a given model resolution and vertical structure he has calculated errors for a range of the values of m, in case of the terrain slope and an atmosphere at rest considered previously by Phillips (1974) and representing an approximation to the standard atmosphere. The obtained errors were generally rather small except at the uppermost layer, centered at the pressure of 300 mb. At that layer the error was quite large for m=0 (when it amounted to the geopotential increment between neighbouring grid points of more than 150 m^2s^{-2}); it then decreased with increasing values of m, and increased again when m was increased beyond the value of 1.2. Since the integer values of m were advantageous for reasons of economy, Janjić has chosen the value m=1 as a compromise between the two requirements, that for minimization of the error and that for computational efficiency.

It is actually readily seen that with temperature having its typical tropospheric lapse rate one should expect the errors of the scheme (7.6) for m=1 to be smaller than those for m=0. To this end, compare the hydrostatic equation (1.8) for $\zeta = \ln p$

$$\frac{\partial \phi}{\partial \ln p} = - RT, \tag{7.7}$$

against that for $\zeta = \ln^2 p$

$$\frac{\partial \phi}{\partial \ln^2 p} = - \frac{RT}{2 \ln p} \quad . \tag{7.8}$$

With tropospheric lapse rates, the slope of the geopotential profile defined by the discretization of (7.8) will be less variable than that defined by the discretization of (7.7), since on the right-hand side of (7.8) temperature is divided by another function which decreases with height at a rate of about the same order of magnitude. Thus, the pressure gradient force errors associated with (7.8) will be smaller, as calculations have indeed shown.

Finally, let us briefly consider the pressure gradient force error in case of isentropic coordinates. Choosing $\eta = \theta$, (2.5) leads to

$$- \nabla_p \phi = - \nabla_\theta (\phi + c_p T) \quad . \tag{7.9}$$

Thus, in return for accepting the inconveniences of the lower boundary condition, in addition to other advantages, pressure gradient force problems due to mountain slope have been avoided. For example, there are no difficulties in simulating a resting atmosphere. Furthermore, with the pressure gradient force being a potential vector, its difference formulation can hardly be chosen in a hydrostatically inconsistent way. However, note that the pressure gradient force problems due to the slope of coordinate surfaces have not changed, in the sense that the condition (4.23) still has to be observed to maintain hydrostatic consistency. Otherwise, large errors are possible.

8. VERTICAL INTERPOLATION OF INITIAL PRESSURE GRADIENT FORCE

Preceding considerations show that with sloping coordinate surfaces it appears not possible to construct a scheme able to simulate an arbitrary resting atmosphere without creating false pressure gradient forces. With a carefully chosen scheme the obtained pressure gradient force errors should be generally rather small; but in special situations (e.g. sharp inversions) large errors are possible.

The approach used so far assumed initial analysis of geopotential, or temperature. If this analysis were done on constant pressure or other surfaces not used as model coordinate surfaces, it would be followed by a vertical interpolation of the analysed variable to obtain its values at model coordinate surfaces. Having the values of geopotential on coordinate surfaces, the initial temperature would be calculated using the hydrostatic equation of the model. If the temperature were interpolated, geopotential would be calculated in that way. Finally, the initial pressure gradient force would be evaluated.

It was pointed out by Sundqvist (1976) that a different sequence of initialization steps is possible. Following an analysis on pressure surfaces, pressure gradient force can be vertically interpolated to obtain its values on model coordinate surfaces. Assuming pressure gradient force to be known on coordinate surfaces, a set of elliptic equations can be derived, and the obtained numerical values used to solve this set of equations for temperature.

Apart from the problems of the analysis and the vertical interpolation error, the method proposed by Sundqvist appears to offer the possibility of simulating an arbitrary resting state without creating false motions due to the pressure gradient force errors. Rather than the initial pressure gradient force, it is the initial temperature that in this case would have an error. This error would be exactly such as to enable the correct initial pressure gradient force field to be recovered by the pressure gradient force scheme of the model; hopefully, at the same time, it would never be so large as to make the temperature field unreasonable.

Unfortunately, calculation of temperature using this approach is associated with difficulties. To derive his equation for temperature Sundqvist has applied the $\nabla_\sigma \cdot$ operator on the pressure gradient force. Had he applied the $\underline{k} \cdot \nabla_\sigma \times$ operator instead he would have obtained a different equation, generally giving a different solution for temperature.

A simpler method of solving for temperature has been used by Mihailovic (1981). He has directly solved the finite-difference pressure gradient force expression for temperature, starting from known value of temperature at the boundaries. This, however, did not avoid the non-uniquenes problem: starting from two boundary points of a grid line two generally different temperature values are obtained at each interior point. In the two-dimensional case, at each point two more values are obtained solving for temperature along the other set of grid lines. Furthermore, each solution obtained along a grid line starting from a boundary point at its one end violates the prescribed temperature at the other end of that line.

The following simple explanation of the non-uniqueness problem has been put forward by Janjic. Namely, let us for simplicity consider the one-dimensional problem. In this case the pressure gradient force contributions in between a grid point and two adjacent grid points are, in general, defined on two different ζ^* surfaces. Thus, at the central point, we shall have two values of ϕ^*. This situation is schematically represented in Fig.14. Since, however, geopotential may not be a linear function of ζ, a unique value of $\delta_\zeta \phi$ cannot yield both values of ϕ^*. Instead we obtain two values of $\delta_\zeta \phi$ and, consequently, two temperatures. In the two-dimensional case two additional values will appear because the problem recurs in the direction of the other coordinate axis.

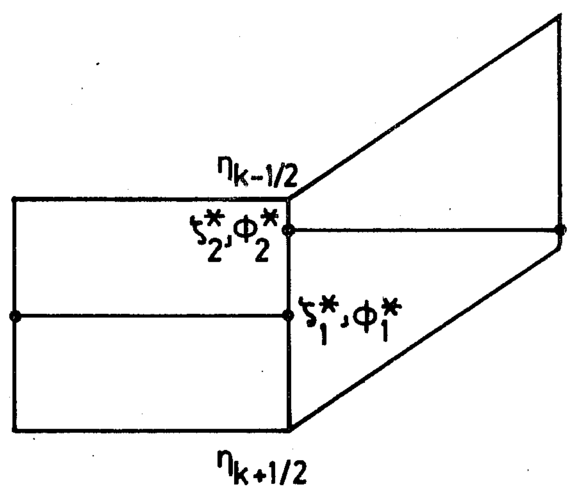

Fig. 14 Schematic illustration of the non-uniqueness problem.

In order to minimize the deviation of the initial eta system pressure gradient force from the pressure gradient force obtained by interpolation from the p system, Janjić and Nicovic (unpublished) used least squares fitting to define a unique linear geopotential profile within each eta (sigma) layer. This profile is then used to calculate unique temperature. However, a more elaborate variational approach is certainly also possible.

9. BLOCKING TECHNIQUES FOR REPRESENTATION OF MOUNTAINS

In addition to the pressure gradient fore problem, terrain-following coordinate systems have other difficulties. Dynamical processes may be distorted by the irregularities inherent to a sigma-type grid (Sadourny et al. 1981). Several of the additional numerical problems (e.g., Simmons and Burridge, 1981; Simmons and Strüfing, 1981), while not fundamental, certainly are uncomfortable. Thus, it has recently a number of times been suggested that alternate approaches to the representation of mountains in numerical models may deserve more attention.

The pressure gradient force problem and to a large extent also the additional numerical problems result from the slope of coordinate surfaces. A number of definitions of the vertical coordinate go to and eventually become equal to pressure as the height is increased. Unfortunately, such coordinates obviously can be only of a limited help in alleviating the pressure gradient force problem, since the reduction of the slope of coordinate surfaces is limited.

A more drastic approach has been used by Egger (1972). The Egger's method consisted of vertical walls, in a sigma system model, placed so as to block the flow in a given sigma layer or layers and thus simulate the barrier effect of steep mountains. Even though remarkably successful, this method had obvious imperfections in not being able to properly accommodate the three-dimensional geometry of steep mountains, and in having steep mountains change their elevation as a function of time.

It has been pointed out by Mesinger (1983) that these two weakneses can be removed by constructing model mountains so that they consist of three-dimensional grid boxes (Fig. 15)

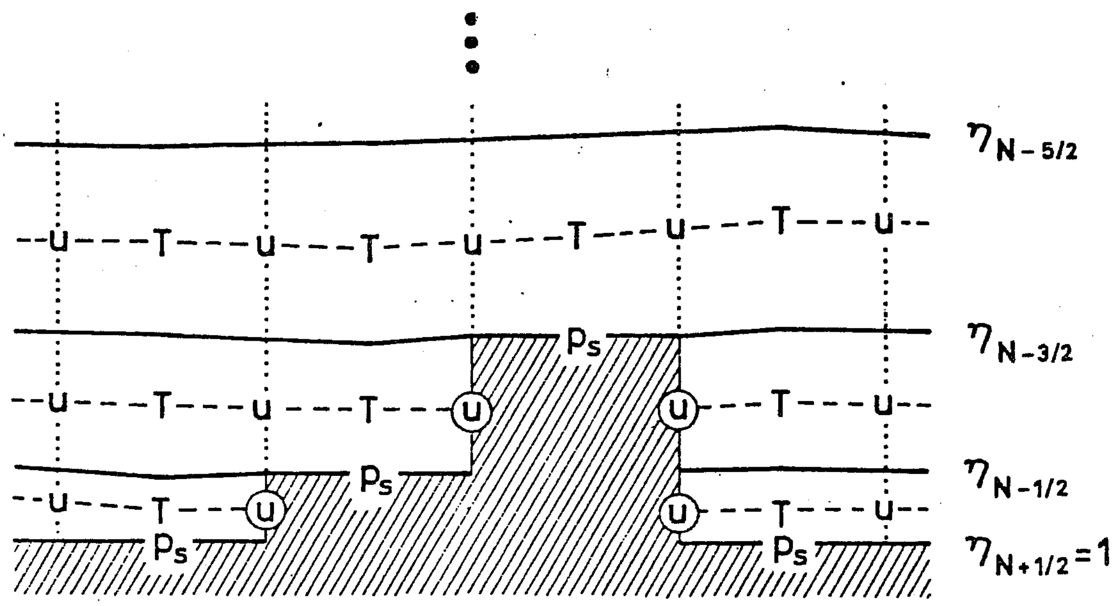

Fig. 15 The vertical grid and indexing used by Mesinger (1983).

and by defining the vertical coordinate in such a way as to have its surfaces remain at fixed elevations at places where they touch (and define) the ground surface. He has pointed out that this can be achieved by the coordinate

$$\eta \equiv \frac{p - p_T}{p_s - p_T} \frac{p_{rf}(z_s) - p_T}{p_{rf}(0) - p_T} \qquad (9.1)$$

Here $p_{rf}(z)$ is a suitably defined reference pressure as a function of z, the geometric height. To have, as stated, mountains formed of the grid boxes, the values of z_s are permitted to take only values given by:

$$\eta_{k+\frac{1}{2}} = \frac{p_{rf}(z_s) - p_T}{p_{rf}(0) - p_T}, \quad k = 1,2,\ldots, N, \tag{9.2}$$

that is, by the values of η chosen for the interfaces of the η layers of the model.

Note that (9.1) implies that

$$\eta = 0 \quad \text{when} \quad p = p_T$$

and

$$\eta = 1 \quad \text{when} \quad z = 0.$$

Furthermore,

$$\eta = (p - p_T)/(p_{rf}(0) - p_T) \quad \text{when} \quad p_s = p_{rf}(z_s).$$

Thus, when pressure is a function of z only, and up to a level which includes the highest model mountains this function is equal to the chosen reference pressure, η coordinate suraces will all be horizontal. Then, for any reasonable choice of the finite-difference scheme, there will be no error in the pressure gradient force. This is a feature which with sigma-type coordinates we saw as possible to achieve only under much more restrictive conditions. But, of course, also in other cases the slope of coordinate surfaces defined by (9.1) will be small, and errors of the pressure gradient force should be very much reduced.

The proposed system makes an effort to preserve, to the maximum extent possible, simplicity in the specification of the boundary condition. As a result, velocity components normal to the ground surface are readily set to zero. Further details of the wall boundary condition may, of course, be a matter of some complexity if particular features of the horizontal differencing are to be maintained. This obviously is a task to be considered for each scheme separately.

The coordinate of the type (9.1) is because of its dependence on z and because the top η surface is permitted to be also at a level p=const > 0 more general than the η coordinate considered by Simmons and Burridge (1981). These two differences, as can be verified following Kasahara (1974), in themselves do not affect the continuous equations. There are, however, differences introduced by the step-like ground surface. One is a simple point of noting that in the pressure tendency equation the vertical integration is to be performed from top to the lowermost value of η, η_s, now not necessarily 1. (And, furthermore, to the lowermost of the values of η_s, along the sides of mountains).

Another difference is the need to consider horizontal discontinuities in $(p_{rf}(z_s) - p_T)/(p_s-p_T)$, resulting from dicontinuities in ground elevation. Thus, the η surfaces, as defined by (9.1) will in fact also be discontinuous.

This difficulty, as also pointed out by Mesinger, can be overcome by assuming that the values of $(p_{rf}(z_s) - p_T)/(p_s-p_T)$ in (9.1) are not the actual values, but rather values interpolated in such a way as to achieve the continuity of η surfaces. An additional property to be required of the interpolation algorithm is that these interpolated values tend to the actual values as the distance from the horizontal sides of the ground surface approaches zero; this is necessary in order to maintain the property of η surfaces to stay fixed to (and define) the horizontal sides of model mountains. It is not difficult to construct an interpolation algorithm having these two properties; however, there is no need to actually do it here, since the details of this algorithm will not be needed by the differencing scheme. The finite-difference scheme, in fact, can be considered to imply an interpolation algorithm with the very same properties.

Consequently, following Kasahara (1974) and Simmons and Burridge (1981), the governing equations for frictionless and adiabatic motion can be written down as follows.

$$\frac{d\underline{V}}{dt} + f\,\underline{k} \times \underline{V} + \nabla\phi + \frac{RT}{p}\nabla p = 0 \,, \qquad (9.3)$$

$$\frac{dT}{dt} - \frac{\kappa T\omega}{p} = 0, \qquad (9.4)$$

$$\frac{\partial}{\partial\eta}\left(\frac{\partial p}{\partial t}\right) + \nabla \cdot \left(\underline{v}\,\frac{\partial p}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\dot{\eta}\,\frac{\partial p}{\partial\eta}\right) = 0 \,, \qquad (9.5)$$

$$\frac{\partial\phi}{\partial\eta} = -\frac{RT}{p}\frac{\partial p}{\partial\eta}\,, \qquad (9.6)$$

$$\omega \equiv \frac{dp}{dt} = -\int_{o}^{\eta} \nabla \cdot \left(\underline{v}\,\frac{\partial p}{\partial\eta}\right) d\eta + \underline{v}.\nabla p \,, \qquad (9.7)$$

$$\frac{\partial p_s}{\partial t} = -\int_{o}^{\eta_s} \nabla \cdot \left(\underline{v}\,\frac{\partial p}{\partial\eta}\right) d\eta \,, \qquad (9.8)$$

$$\dot{\eta}\,\frac{\partial p}{\partial\eta} = -\frac{\partial p}{\partial t} - \int_{o}^{\eta} \nabla \cdot \left(\underline{v}\,\frac{\partial p}{\partial\eta}\right) d\eta \,. \qquad (9.9)$$

The subscripts of the del operator have here been omitted; and, boundary conditions were assumed p=const at the top boundary $\eta=0$, and $\dot{\eta}=0$ at $\eta=0$ and at horizontal parts of the ground surface, $\eta=\eta_s$.

The blocking approach, as outlined here, has not yet been tested in actual integrations using comprehensive atmospheric models. However, two groups are presently involved in developing models of this type; thus, some experience in the performance of such models is likely to be available in a reasonably short time.

10. CONSERVATION OF ANGULAR MOMENTUM

For an analysis of the angular momentum and also the energy conservation feature we shall here use the notation of Eqs. (9.3) to (9.9) and Fig.15 of the preceding section. Note that these equations did not actually depend on the specific definition of eta apart from the stated boundary conditions, which are the same as those used with terrain following coordinates. Thus, this system as well as the figure can be considered as rather general, covering both the terrain following coordinates, which will have no step mountains, as well as the system including the presence of step mountains in case that eta is defined to be a coordinate of the type (9.1). We shall here present the analysis of Mesinger (1983), which in turn to a very large degree follows that of Simmons and Burridge (1981, hereafter referred to as SB; also Simmons and Strüfing, 1981). In addition to permitting a more general choice of the vertical coordinate, it departs from their analysis i) in not making a decision of the finite-difference analog of the hydrostatic equation; and ii) in considering, from the start of the analysis, the effects of the horizontal discretization along with those of the vertical discretization - since in the case of the step mountains a separation of these two discretization effects would not seem appropriate. To save space, Mesinger has restricted his consideration of the horizontal discretization to longitude only. Further generalization to two horizontal coordinates should in most cases present no difficulties.

In the continuous case, we want to evaluate the integral of the pressure gradient force with respect to pressure, and longitude, λ,

$$- \int_0^{2\pi} \int_0^{\eta_s} \left(\frac{\partial \phi}{\partial \lambda} + \frac{RT}{p} \frac{\partial p}{\partial \lambda}\right) \frac{\partial p}{\partial \eta} \, d\eta \, d\lambda \ , \qquad (10.1)$$

having in mind a possible existence of step-like mountains as shown in Fig.15. After use is made of

$$\frac{\partial \phi}{\partial \lambda} \frac{\partial p}{\partial \eta} = \frac{\partial}{\partial \lambda} \left(\phi \frac{\partial p}{\partial \eta}\right) - \frac{\partial}{\partial \eta} \left(\phi \frac{\partial p}{\partial \lambda}\right) + \frac{\partial \phi}{\partial \eta} \frac{\partial p}{\partial \lambda} \ ,$$

of the boundary condition p = const along the surface $\eta = 0$, and of (9.6), we obtain

$$\int_s \phi \frac{\partial p}{\partial s} \, ds \ . \qquad (10.2)$$

Here s is defined to follow the ground surface at the considered latitude, increasing eastward; and up the western and down the eastern sides of step mountains in case a choice of eta of the type (9.1) is being used. Thus, the contribution of the pressure gradient force to the change of angular momentum will be a "mountain torque" term proportional to (10.2).

To have no false production of angular momentum due to the finite-difference scheme, the analog to (10.1) has to be chosen so that it results in an analog to (10.2). Following SB, the analog to (10.1) is considered to be of the form

$$- \sum_{u} \left[\delta_{\lambda}\phi_k + \left(\frac{RT}{p}\frac{\partial p}{\partial \lambda}\right)_k \right] \overline{\Delta p_k}^{\lambda} \; \Delta\lambda \; . \qquad (10.3)$$

The summation here refers to all u points having u as a time-dependent variable; thus, it does not include the mountain side u points. Also as in SB, geopotential is carried at interfaces, and ϕ_k is defined by

$$\phi_k \equiv \phi_{k+\frac{1}{2}} + \alpha_k \, RT_k . \qquad (10.4)$$

This definition would appear to permit a rather general class of the pressure gradient force (and hydrostatic equation) schemes; for example, it includes the family of schemes proposed by Janjić (1977).

Transformation of (10.3) with the help of

$$\delta_{\lambda}(AB) = \overline{A}^{\lambda} \; \delta_{\lambda}B + \overline{B}^{\lambda} \; \delta_{\lambda}A,$$

results in a term with contributions from T points which all cancel except for those from T points next to the mountain sides. Further transformation of the second term obtained in this way, with the help of (10.4) and another general rule

$$A_{k+\frac{1}{2}} \, \delta_{\lambda} \, \Delta B_k = \Delta(A\delta_{\lambda}B)_k - \Delta A_k \delta_{\lambda} B_{k-\frac{1}{2}}$$

similarly gives terms which all cancel except for those from u points next to the ground surface. In this way, it is seen that (10.3) gives

$$\sum_{u,a} \overline{\phi_{k+\frac{1}{2}}}^{\lambda} \delta_\lambda p_{k+\frac{1}{2}} \Delta\lambda - \sum_{T,w} \phi_k \Delta p_k + \sum_{T,e} \phi_k \Delta p_k$$
$$- \sum_{u} \left[\overline{\Delta\phi_k}^{\lambda} \delta_\lambda p_{k-\frac{1}{2}} - \overline{\alpha_k RT_k}^{\lambda} \delta_k \Delta p_k + \left(\frac{RT}{p} \frac{\partial p}{\partial \lambda}\right)_k \overline{\Delta p_k}^{\lambda} \right] \Delta\lambda \tag{10.5}$$

Here the first three pairs of summation subscripts denote that summations are to be performed over all the u points immediately above, T points immediately west, and T points immediately east of the ground surface, respectively. With the grid as shown in Fig.15, the first three terms in (10.5) are in this way readily recognized to represent the simplest possible analog to the mountain torque integral (10.2). Thus, for a given definition of $\Delta\phi_k$ and α_k, the angular momentum conservation will be achieved if within the analog to the pressure gradient force it is chosen

$$\left(\frac{RT}{p} \frac{\partial p}{\partial \lambda}\right)_k = - \frac{1}{\overline{\Delta p_k}^{\lambda}} \left(\overline{\Delta\phi_k}^{\lambda} \delta_k p_{k-\frac{1}{2}} - \overline{\alpha_k RT_k}^{\lambda} \delta_\lambda \Delta p_k \right) , \tag{10.6}$$

so as to make the expression in the square bracket of (10.5) equal to zero.

The requirement (10.6) reduces to that obtained by SB, their Eq. (3.14), for a special choice made by SB of the finite-difference analog to the hydrostatic equation. Thus, the more general definition of the vertical coordinate, permitting _inter alia_ a step-like representation of mountains, has been seen to have had no effect on the angular momentum conservation requirement.

Using the sigma coordinate defined as

$$\sigma \equiv \frac{p - p_T}{\pi} , \quad \pi \equiv p_s - p_T , \tag{10.7}$$

and considering the vertical differencing only, Arakawa and Suarez (1983) show that the vertically integrated circulation feature (1.6) will be maintained when the pressure gradient force is of the form

$$- \nabla\phi_k - \frac{1}{\pi} \left[\phi_k - \frac{\Delta(\sigma\phi)_k}{\Delta\sigma_k} \right] \nabla\pi . \tag{10.8}$$

It can readily be demonstrated that the one-dimensional version of (10.8) is one member of the family of schemes obtained by removing the horizontal discretization from (10.6). Namely, the vertical differencing scheme thus obtained reduces to the one-dimensional version of (10.8) for the special choice of the vertical co-ordinate, (10.7), used by Arakawa and Suarez.

11. CONSERVATION OF ENERGY

The rate of kinetic energy generation by the pressure gradient force, due to eastward motion, in a vertical plane located at latitude ψ and bounded by the surface $\eta=0$ and by longitudes λ_1,λ_2, per unit distance in meridional direction is equal to

$$-\frac{1}{g}\int_{\lambda_1}^{\lambda_2}\int_{o}^{\eta_s}\frac{u}{a\cos\psi}\left(\frac{\partial\phi}{\partial\lambda}+\frac{RT}{p}\frac{\partial p}{\partial\lambda}\right)\frac{\partial p}{\partial\eta}\,d\eta\,d\lambda\;. \tag{11.1}$$

Here u is the eastward velocity component, and a the radius of the earth. As it was done in the preceding section, this consideration is also restricted to one horizontal coordinate only.

The geopotential gradient part of the integrand of (11.1) can with the help of the two-dimensional version of (9.5), and (9.6), be transformed into

$$\frac{1}{a\cos\psi}\frac{\partial}{\partial\lambda}\left(u\phi\frac{\partial p}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left[\phi\left(\frac{\partial p}{\partial t}+\dot{\eta}\frac{\partial p}{\partial\eta}\right)\right]+\frac{RT}{p}\left(\frac{\partial p}{\partial t}+\dot{\eta}\frac{\partial p}{\partial\eta}\right)\frac{\partial p}{\partial\eta} \tag{11.2}$$

The first term here integrates to zero for a closed region, and for the present boundary condition. With p=const at the top boundary $\eta=0$, and $\dot{\eta}=0$ at the top as well as at the lower boundary, the second term integrates to give a contribution

$$-\frac{1}{g}\int_{\lambda_1}^{\lambda_2}\phi_s\frac{\partial p_s}{\partial t}\,d\lambda. \tag{11.3}$$

The third term of (11.2) and the pressure gradient part of (11.1) are compensated by the terms arising from the thermodynamic equation

$$\frac{1}{g}\int_{\lambda_1}^{\lambda_2}\int_{o}^{\eta_s} c_p \frac{\kappa T\omega}{p}\frac{\partial p}{\partial \eta}\, d\eta\, d\lambda = \frac{1}{g}\int_{\lambda_1}^{\lambda_2}\int_{o}^{\eta_s} \frac{RT}{p}\left(\frac{\partial p}{\partial t} + \dot{\eta}\frac{\partial p}{\partial \eta} + \frac{u}{a\cos\psi}\frac{\partial p}{\partial \lambda}\right)\frac{\partial p}{\partial \eta}\, d\nu\, d\lambda . \tag{11.4}$$

To have no false production of energy in transformations between the kinetic and total potential energy this compensation has to be achieved in the discrete case as well.

In parallel with (10.3), we consider the analog to (11.1) to be of the form

$$-\frac{1}{g}\sum_{u} \frac{u_k}{a\cos\psi}\left[\delta_\lambda \phi_k + \left(\frac{RT}{p}\frac{\partial p}{\partial \lambda}\right)_k\right]\overline{\Delta p_k}^{\lambda}\,\Delta\lambda \ . \tag{11.5}$$

As it is to be used to determine an analog to $\kappa T\omega/p$, carried at T points, we transform each of its two parts to a sum over T points. Considering the geopotential gradient part, note that

$$\sum_{u} U\ \delta_\lambda P = -\sum_{T} P\ \delta_\lambda U$$

for variables U and P defined at u and T points, respectively, of a staggered grid a shown in Fig.15, bounded by u points at which U=0. Thus, the geopotential gradient part of (11.5) is transformed into

$$\frac{1}{g}\sum_{T}\frac{\phi_k}{a\cos\psi}\,\delta_\lambda\,(u_k\overline{\Delta p_k}^{\lambda})\,\Delta\lambda, \tag{11.6}$$

which accomplishes the equivalent of integrating to zero the first term of (11.2). Further transformation of (11.6) in the manner of (11.2) requires use of the finite-difference analog of the continuity equation. We choose

$$\Delta W_k + \frac{1}{a\cos\psi}\,\delta_\lambda\,(u_k\overline{\Delta p_k}^{\lambda}) = 0 \quad , \tag{11.7}$$

where, for brevity,

$$W \equiv \frac{\partial p}{\partial t} + \dot{\eta}\frac{\partial p}{\partial \eta} \ .$$

Making, in addition, use of

$$A_k \, \Delta B_k = \Delta(AB)_k + (A_{k-\frac{1}{2}} - A_k)B_{k-\frac{1}{2}} + (A_k - A_{k+\frac{1}{2}})B_{k+\frac{1}{2}},$$

and of the boundary conditions on p and on $\dot{\eta}$, we see that (11.6) is transformed into

$$-\frac{1}{g}\sum_{p_s} \phi_s \frac{\partial p_s}{\partial t} \Delta\lambda - \frac{1}{g}\sum_T \left[(\phi_{k-\frac{1}{2}} - \phi_k) W_{k-\frac{1}{2}} + (\phi_k - \phi_{k+\frac{1}{2}})W_{k+\frac{1}{2}}\right]\Delta\lambda \quad (11.8)$$

containing, as its first term, an analog to (11.3).

Writing the analog to (11.4) as

$$\frac{1}{g}\sum_T c_p \left(\frac{\kappa T\omega}{p}\right)_k \Delta p_k \, \Delta\lambda \, , \quad (11.9)$$

we recognize that the expression in the square bracket of (11.8) is to define one ("vertical") part of the analog to $\kappa T\omega/p$. This part

$$\frac{1}{c_p \Delta p_k} \left[(\phi_{k-\frac{1}{2}} - \phi_k)W_{k-\frac{1}{2}} + (\phi_k - \phi_{k+\frac{1}{2}})W_{k+\frac{1}{2}}\right] , \quad (11.10)$$

making use of (10.4) and (11.7) we see it can also be written as

$$\frac{1}{c_p \, \Delta p_k a \cos\psi} \left[\Delta\phi_k \sum_{r=1}^{k-1} \delta_\lambda (u_r \overline{\Delta p_r}^\lambda) - \alpha_k RT_k \delta_\lambda (u_k \overline{\Delta p_k}^\lambda)\right] . \quad (11.11)$$

Regarding the pressure gradient part of (11.5), note that, again with U=0 at boundary points, we have

$$\sum_u U\overline{P}^\lambda = \sum_T P\overline{U}^\lambda .$$

Thus, it is found that the required cancellation will be achieved if the remaining ("horizontal") part of the analog to $\kappa T\omega/p$ is chosen to be of the form

$$\frac{1}{c_p \, a \cos\psi} \, \overline{u_k \left(\frac{RT}{p} \frac{\partial p}{\partial \lambda}\right)_k}^\lambda , \quad (11.12)$$

where the analog to $(RT/p)\, \partial p/\partial\lambda$ is the same as that used in the momentum equation.

Alternatively, if this analog to $(RT/p)\partial p/\partial\lambda$ is restricted to the usual sigma system form

$$R\overline{T}_k^{\lambda}\ \left(\frac{1}{p}\frac{\partial p}{\partial\lambda}\right)_k ,$$

as chosen by SB, the transformation to the sum over T points can be done so as to obtain, instead of (11.12),

$$\frac{\kappa T_k}{\Delta p_k a\ \cos\psi}\ \overline{u_k\ \overline{\Delta p}_k^{\lambda}\ \left(\frac{1}{p}\frac{\partial p}{\partial\lambda}\right)_k}^{\lambda} . \tag{11.13}$$

Thus, it is seen that in this case some freedom remains in the choice of the analog to $\kappa T\omega/p$.

The result (11.10)/(11.11), and (11.12) or (11.13), has no differences compared to that given by SB except for those reflecting the absence of a specific choice of the hydrostatic equation and the inclusion of the horizontal differencing in the analysis given here.

Restricting the pressure gradient force to their result (10.8), Arakawa and Suarez find that in order to conserve energy $\kappa T\omega/p$ in the thermodynamic equation must be of the form

$$\frac{1}{c_p\pi}\ \Big[\Big(\phi_k - \frac{\Delta(\sigma\phi)_k}{\Delta\sigma_k}\Big)\Big(\frac{\partial}{\partial t} + \underline{V}_k\cdot\ \nabla\Big)\pi$$

$$+ \frac{1}{\Delta\sigma_k}\ \Big((\pi\dot{\sigma})_{k+\frac{1}{2}}(\phi_k - \phi_{k+\frac{1}{2}}) + (\pi\dot{\sigma})_{k-\frac{1}{2}}(\phi_{k-\frac{1}{2}} - \phi_k)\Big)\Big] . \tag{11.14}$$

As in the previous section, it is readily demonstrated that the one-dimensional version of (11.14) is a special case of the scheme obtained by removing the horizontal discretization from the sum of (11.10) and (11.12) or (11.13). The scheme thus obtained reduces to the one-dimensional version of (11.14) after insertion of the vertical coordinate (10.7), used by Arakawa and Suarez.

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