Applications of Generalized Stability Theory to Deterministic and Statistical Prediction

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1 Deterministic Predictability

The atmospheric variables in a forecast model are specified by the finite dimensional state vector ϕ which evolves according to the deterministic equation:

$$\frac{d\phi}{dt} = f(\phi) \ . \tag{1}$$

Consider a solution of the forecast equations $\phi(t)$ starting from a given initial state. Sufficiently small forecast errors $\psi \equiv \delta\phi$ are governed in the linear approximation by the tangent linear equations:

$$\frac{d\psi}{dt} = \left.\frac{\partial f}{\partial \phi}\right|_{\phi(t)} \psi \ . \tag{2}$$

The Jacobian matrix $\mathbf{A}(t) = \frac{\partial f}{\partial \phi}$ is evaluated along the known trajectory $\phi(t)$ and is considered to be known.

The matrix $\mathbf{A}(t)$ is time dependent and its realizations in general do not commute at different times, i.e. $\mathbf{A}(t_1)\mathbf{A}(t_2) \neq \mathbf{A}(t_2)\mathbf{A}(t_1)$. It follows that the evolution of the error field can not be determined from analysis of the eigenvalues and eigenfunctions of $\mathbf{A}$, as would be the case for time independent normal matrices. A normal matrix commutes with its hermitian transpose and has orthogonal eigenfunctions and the analysis must be made using the methods of Generalized Stability Theory (GST) (cf for a review Farrell & Ioannou , 1996a). GST concentrates attention on the behavior of the propagator $\Phi(t,0)$ which is the matrix that transforms the initial error $\psi(0)$ to the error $\psi(t)$ at time t:

$$\psi(t) = \Phi(t,0) \ \psi(0) \ . \tag{3}$$

Once the matrix $\mathbf{A}(t)$ of the tangent linear system is available the propagator is readily calculated. Consider a piecewise approximation of the continuous operator $\mathbf{A}(t)$: $\mathbf{A}(t) = \mathbf{A}_i$ where $\mathbf{A}_i$ is the mean of $\mathbf{A}(t)$ over $(i-1)\tau \le t < i\tau$ for small enough τ. At time $t = n\tau$ the propagator is given by

$$\Phi(t,0) = \prod_{i=1}^{n} e^{\mathbf{A}_i \tau} \ . \tag{4}$$

If $\mathbf{A}$ is autonomous (time independent) the propagator is the matrix exponential

$$\Phi(t,0) = e^{\mathbf{A}t} \ . \tag{5}$$

Deterministic predicability is limited by the optimal growth in the error over the interval $[0,t]$:

$$||\Phi(t,0)|| \equiv \max \frac{||\psi(t)||}{||\psi(0)||} \ . \tag{6}$$

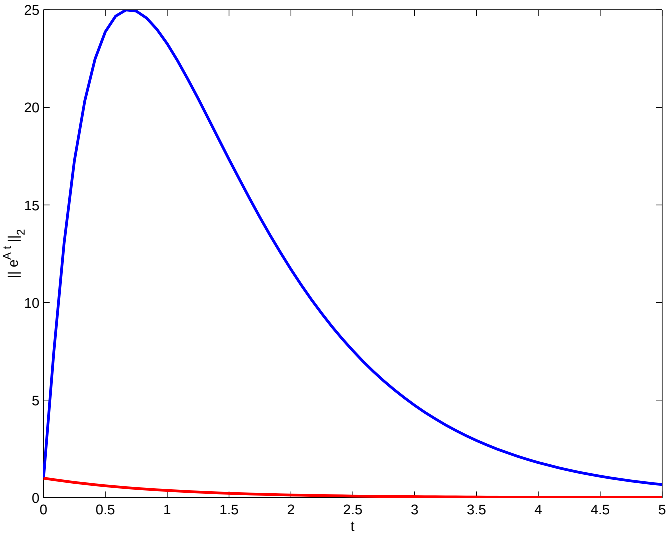

Figure 1: The upper curve gives the optimal growth as function of time for the simple matrix (7). The optimal growth is given by the norm of the propagator $e^{\mathbf{A}t}$. The lower curve shows the evolution of the amplitude of the least damped eigenmode which decays at a rate of -1.

This maximization is over all initial errors $\psi(0)$. The optimal growth for each t is the norm of the propagator $||\Phi(t,0)||$. Implicit in the definition of the optimal is the choice of the norm. In most application $||\psi(t)||^2$ is chosen to correspond to the total perturbation energy (for a discussion on the choice of the inner product see Palmer et al, 1998; for the use of norms that do not derive from inner products see Farrell & Ioannou, 2000).

In order to illustrate GST we apply it to the simple autonomous Reynolds[1] matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} -1 & 100 \\ 0 & -2 \end{pmatrix} \quad . \tag{7}$$

Consider the growth in the tangent linear system:

$$\frac{d\psi}{dt} = \mathbf{A}\psi \quad . \tag{8}$$

Traditional stability theory concentrates on the growth associated with the most unstable mode which in this example gives decay at rate -1 indicating that the error decays exponentially at this rate. While this is indeed the case for very large times, the optimal error growth, shown by the upper curve in Fig. 1, is much greater at all times than that predicted by the fastest growing mode (the lower curve in Fig. 1). The modal prediction fails to capture the error growth because $\mathbf{A}$ is non-normal i.e. $\mathbf{A}\mathbf{A}^\dagger \neq \mathbf{A}^\dagger\mathbf{A}$ and its eigenfunctions are non-orthogonal.

The optimal growth is calculated as follows:

$$G = \frac{||\psi(t)||^2}{||\psi(0)||^2} = \frac{\psi(t)^\dagger\psi(t)}{\psi(0)^\dagger\psi(0)} = \frac{\psi(0)^\dagger e^{\mathbf{A}^\dagger t} e^{\mathbf{A}t}\psi(0)}{\psi(0)^\dagger\psi(0)} \quad . \tag{9}$$

This Rayleigh quotient reveals that the maximum eigenvalue of the positive definite matrix $e^{\mathbf{A}^\dagger t}e^{\mathbf{A}t}$ determines the square of the optimal growth at time t. The corresponding eigenvector is the initial perturbation that leads to this growth, called the optimal perturbation (Farrell, 1988). Alternatively, we can proceed with a Schmidt decomposition (singular value decomposition) of the propagator:

$$e^{\mathbf{A}t} = \mathbf{U}\Sigma\mathbf{V}^\dagger \tag{10}$$

[1]It is called the Reynolds matrix because it captures the emergence of rolls in three dimensional boundary layers that are responsible for transition to turbulence.

with $\mathbf{U}$ and $\mathbf{V}$ unitary matrices and Σ a diagonal matrix with elements the singular values, σ_i, of the propagator which give the growth achieved at time t by each of the orthogonal columns of $\mathbf{V}$. The largest singular value is the optimal growth and the corresponding column of $\mathbf{V}$ is the optimal perturbation. The orthogonal columns, v_i, of $\mathbf{V}$ are called optimal vectors (or singular vectors), and the orthogonal columns, u_i, of $\mathbf{U}$ are the evolved optimal vectors (or evolved singular vectors) because from the Schmidt decomposition we have

$$\sigma_i u_i = e^{\mathbf{A}t} v_i \ . \tag{11}$$

The forecast system has typical dimension 10^7 so we can not calculate the propagator directly as in (4) in order to obtain the optimal growth. Instead we integrate the system:

$$\frac{d\psi}{dt} = \mathbf{A}\psi \tag{12}$$

forward to obtain $\psi(t) = e^{\mathbf{A}t}\psi(0)$ (or its equivalent in a time dependent system), and then integrate the adjoint system:

$$\frac{d\psi}{dt} = -\mathbf{A}^\dagger \psi \tag{13}$$

backward in order to obtain $e^{\mathbf{A}^\dagger t}\psi(t) = e^{\mathbf{A}^\dagger t} e^{\mathbf{A}t}\psi(0)$. We can then find the optimal vectors (singular vectors) by the power method (cf Moore & Farrell, 1993). The leading optimal vectors are used as a basis for the ensemble members in the ensemble forecast at ECMWF because they span the perturbation structures that can lead to appreciable growth of the forecast error (cf Gelaro, et al, 1998).

We have already remarked that the optimal growth depends on the norm. This is expected because physical quantities are norm-dependent. While for autonomous operators there are ”normal coordinates ” in which the operator is rendered formally normal, this coordinate system is neither physical nor generic because time dependent operators, such as the tangent linear forecast system, are inherently non-normal, in the sense that there is no coordinate transformation that can render a general $\mathbf{A}(t)$ simultaneously normal at all times. Analysis of the error growth in time dependent systems necessarily proceeds by analysis of the propagator in the manner outlined above.

The tangent linear system is asymptotically unstable in the sense that the Lyapunov exponent of the tangent linear system is positive. Lyapunov showed that for a general class of time dependent but bounded matrices $\mathbf{A}(t)$ the perturbations $\psi(t)$ grow at most exponentially at large enough times and $\psi(t) \approx e^{\lambda t}$ as $t \to \infty$. The Lyapunov exponent of any tangent linear system can be calculated by evaluating:

$$\lambda = \lim_{t\to\infty} \frac{\ln ||\psi(t)||}{t} \ , \tag{14}$$

This asymptotic measure of loss of predictability is independent of the norm.

It is of interest and of practical importance to determine the perturbation subspace that supports this asymptotic exponential magnification of errors. Because this subspace has a much smaller dimension than that of the tangent linear system a theory that characterizes this subspace can lead to economical truncations of the tangent linear system. Such a truncation would be useful in advancing the error covariance of the tangent linear system which is required for optimal state estimation. We will argue that the inherent non-normality of time dependent operators is the source of the exponential increase of forecast errors and the key to understanding the error growth.

Consider a harmonic oscillator with frequency ω in normal coordinates (i.e. energy coordinates) $\psi = [\omega x, v]^T$, where x is the displacement and $v = \dot{x}$. The system is governed by:

$$\frac{d\psi}{dt} = \mathbf{A}\psi \tag{15}$$

with

$$\mathbf{A} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{16}$$

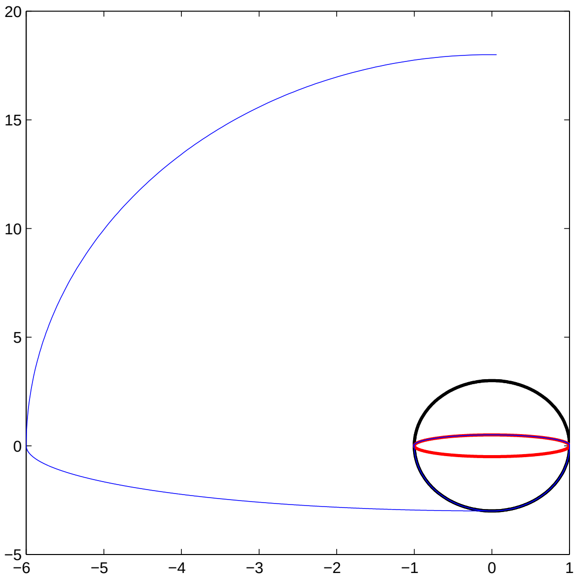

Figure 2: The parametric instability of the harmonic oscillator is caused by the non-normality of the time dependent operator. So long as there is instantaneous growth and the time dependent operators do not commute asymptotic exponential growth occurs.

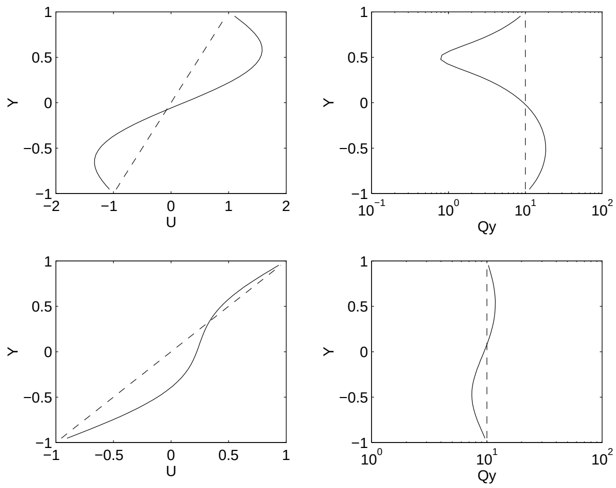

Figure 3: Left panels : the mean flow velocity as a function of latitude for the Rayleigh stable example. Right panel: the associated mean vorticity gradient with $\beta = 10$.

This is a normal system $\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}$ and the system trajectory lies on a constant energy surface which is a circle. In these coordinates the perturbation amplitude is the radius of the circle and there is no growth.

Assume now that the frequency switches periodically between ω_1 and ω_2: $\pi/(2\omega_1)$ units of time the frequency is ω_1 and $\pi/(2\omega_2)$ units of time the frequency is ω_2. There is no single transformation of coordinates that renders the matrix $\mathbf{A}$ simultaneously normal when $\omega = \omega_1$ and when $\omega = \omega_2$ and we revert to the state $\psi = [x, v]^T$ with dynamical matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_{1,2}^2 & 0 \end{pmatrix} . \tag{17}$$

When the frequency is ω_1 the state ψ traverses the outer ellipse of Fig. 2 and when the frequency is ω_2 it traverses the inner ellipse of Fig. 2. As can be easily seen from Fig. 2 the trajectory goes to infinity and despite the neutral stability at each instant of time the time dependent system is exponentially unstable. How is that possible? The key lies in the inherent non-normality of the operator in time dependent systems. If the

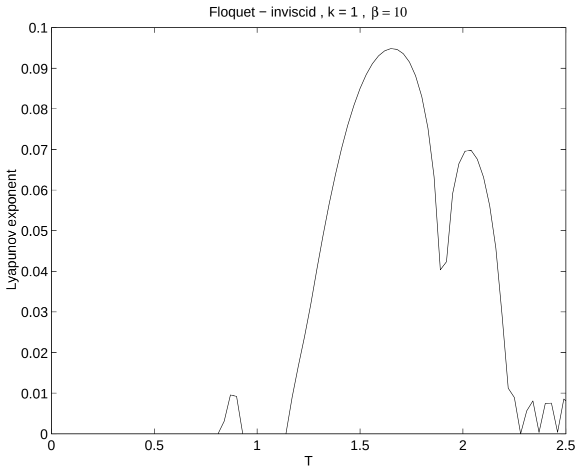

Figure 4: The Lyapunov exponent as a function of the switching period T for the Rayleigh stable example shown in Fig. 3. The time dependent flow results from periodic switching every T time units of the flows shown. The zonal wavenumber is $k=1$ and $\beta = 10$.

operator were time independent and stable transient growth necessarily gives way to eventual decay. In contrast, a time dependent operator can reamplify the perturbations that would have otherwise decayed. This process of continual rejuvenation producing asymptotic destabilization is generic and does not depend on the stability properties of the instantaneous operator state (cf Farrell & Ioannou, 1999).

As an example consider harmonic perturbations $\psi(y,t)e^{ikx}$ on a time dependent mean flow $U(y,t)$ in a β plane channel $-1 \le y \le 1$. The perturbations evolve according to:

$$\frac{d\psi}{dt} = \mathbf{A}(t)\ \psi \tag{18}$$

with time dependent operator:

$$\mathbf{A} = \nabla^{-2}\left(-ik\,U(y,t)\ \nabla^2 - ik\left(\beta - \frac{d^2U(y,t)}{dy^2}\right)\right) . \tag{19}$$

According to Rayleigh's theorem (cf Drazin & Reid, 1981) this flow can not sustain growth unless the mean vorticity gradient $\beta - U''$ changes sign. Let us consider flows which are asymptotically stable at all times and for simplicity that the mean velocity switches periodically between the two flows shown in the left panels of Fig. 3. The corresponding mean vorticity gradient is shown in the right panels of the same figure. Despite the asymptotic stability of each instantaneous flow the periodically varying flow is asymptotically unstable. The Lyapunov exponent of the instability as a function of the switching period is shown in Fig. 4.

This instability arises from sustaining the transient growth of the operator through time dependence. The very same process is operative when the flow is varying continuously in time. Then the Lyapunov exponent can be shown to depend on two parameters: the fluctuation amplitude and the autocorrelation time, T_c, of the fluctuations (cf Farrell & Ioannou, 1999). Snapshots of the perturbation structure revealing the process of accumulation of transient growth by the interaction of the perturbations with the time dependent operator are shown in Fig.5. This mechanism of error growth predicts that the perturbation structure should project most strongly on the subspace of the leading optimal (singular) vectors. This is indeed the case as is demonstrated in Fig. 6.

Study of the asymptotic error structure in more realistic tangent linear systems confirm the conclusions presented above (cf Gelaro et al, 2002). We conclude that error structure in forecast models projects strongly on the optimal vectors of the model. This result is key for dynamical evolution of the error covariance which is required for optimal state estimation of the forecast model (cf Farrell & Ioannou, 2001).

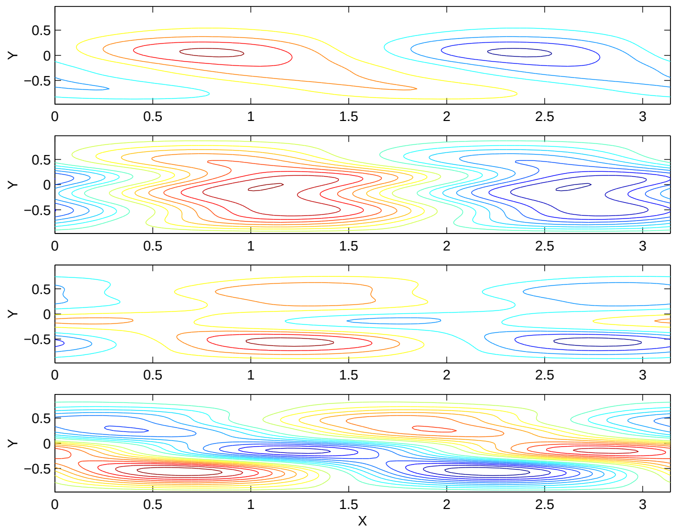

Figure 5: The structure of the Lyapunov vector in the zonal (x), meridional (y) plane at four consecutive times separated by T_c. The r.m.s. velocity fluctuation is 0.16 *and the noise autocorrelation time is* $T_c = 1$*. The zonal wavenumber is* $k = 2$, $\beta = 0$, *and the Reynolds number is* $Re = 800$*. The Lyapunov exponent is* $\lambda = 0.2$*. At first (top panel) the Lyapunov vector is configured to grow producing an increase over* T_c *of* 1.7*, in the next period the Lyapunov vector has assumed a decay configuration (second panel from top) and suffers a decrease of* 0.7*, subsequently (third panel from top) it enjoys a slight growth of* 1.1*, and finally (bottom panel) a growth by* 1.8*.*

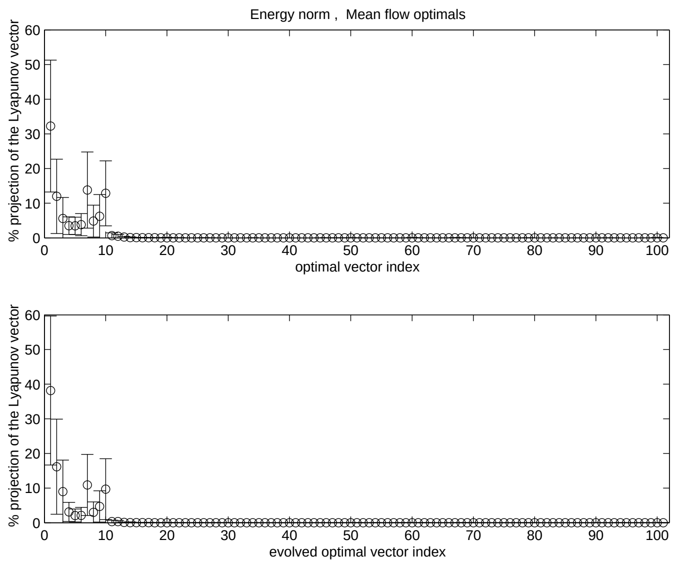

Figure 6: Top panel: Mean projection and standard deviation of the Lyapunov vector on the optimal vectors of the mean flow calculated for a time interval equal to T_c *in the energy norm. Bottom panel: The mean projection and standard deviation of the Lyapunov vector on the* T_c *evolved optimal vectors of the mean flow in the energy norm.*

2 Statistical Predictability of Certain Systems

Consider the perturbation structure, ψ, maintained by the forced equation:

$$\frac{d\psi}{dt} = \mathbf{A}\psi + \mathbf{F}\eta(t) \ . \tag{20}$$

Here $\mathbf{A}$ may be the deterministic linear operator governing evolution of large scale perturbations about the mean midlatitude flow, and $\mathbf{F}\eta(t)$ an additive stochastic forcing with spatial structure $\mathbf{F}$, representing neglected

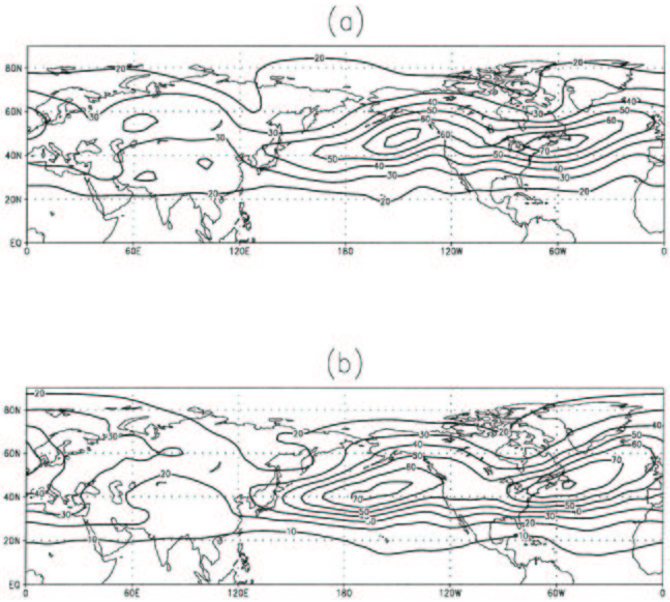

Figure 7: The horizontal structures of the rms of the geopotential height at 350 mb from the linear model (a) and from the perpetual Jan run (b). Contour interval 10 m. From Zhang & Held (1999).

nonlinear terms. For simplicity we assume that $\eta(t)$ is a white noise process with zero mean and unit variance. We wish to determine the perturbation covariance:

$$\mathbf{C}(t) = <\psi\psi^\dagger> \quad , \tag{21}$$

where $<\cdot>$ denotes the ensemble average over the realizations of the forcing $\mathbf{F}\eta(t)$. If a steady state is reached $<\cdot>$ is also the time mean covariance. We argue (Farrell & Ioannou, 1993) that the midlatitude jet climatology can be obtained in this way based on the energetic support of cyclogenesis arising from rapid linear transient growth process associated with the non-normality of $\mathbf{A}$. The diagonal elements of the steady state covariance $\mathbf{C}$ are the climatological variance of ψ, and they locate the storm track regions. Moreover all mean quadratic fluxes are also derivable from $\mathbf{C}$ and we will show that this formulation produces the observed climatological fluxes of heat and momentum. In this way we obtain a theory for the climate and can address systematically statistical predictability questions such as: how to determine the sensitivity of the climate, i.e. of $\mathbf{C}$, to changes in the forcing structure of $\mathbf{F}$, and how to determine the sensitivity of the climate to changes in the mean operator $\mathbf{A}$.

If $\eta(t)$ is a white noise process it can be shown (cf Farrell & Ioannou, 1996) that:

$$\mathbf{C}(t) = \int_0^t e^{\mathbf{A}s}\mathbf{Q}e^{\mathbf{A}^\dagger s}\, ds \, , \tag{22}$$

where

$$\mathbf{Q} = \mathbf{F}\mathbf{F}^\dagger \, , \tag{23}$$

is the covariance of the forcing. It can be also shown that the ensemble mean covariance evolves according to the deterministic equation:

$$\frac{d\mathbf{C}}{dt} = \mathbf{AC} + \mathbf{CA}^\dagger + \mathbf{Q} \equiv \mathscr{A}\mathbf{C} + \mathbf{Q} \, , \tag{24}$$

where $\mathscr{A}$ is a linear operator. It should be noted that the above equation is also valid for non-autonomous $\mathbf{A}(t)$. If $\mathbf{A}$ is time independent the solution of the above equation is:

$$\mathbf{C}(t) = e^{\mathscr{A}t}\,\mathbf{C}(0) + \left(\int_0^t e^{\mathscr{A}(t-s)} ds\right)\mathbf{Q} = e^{\mathscr{A}t}\,\mathbf{C}(0) + \mathscr{A}^{-1}\left(e^{\mathscr{A}t} - \mathscr{I}\right)\mathbf{Q} \; . \tag{25}$$

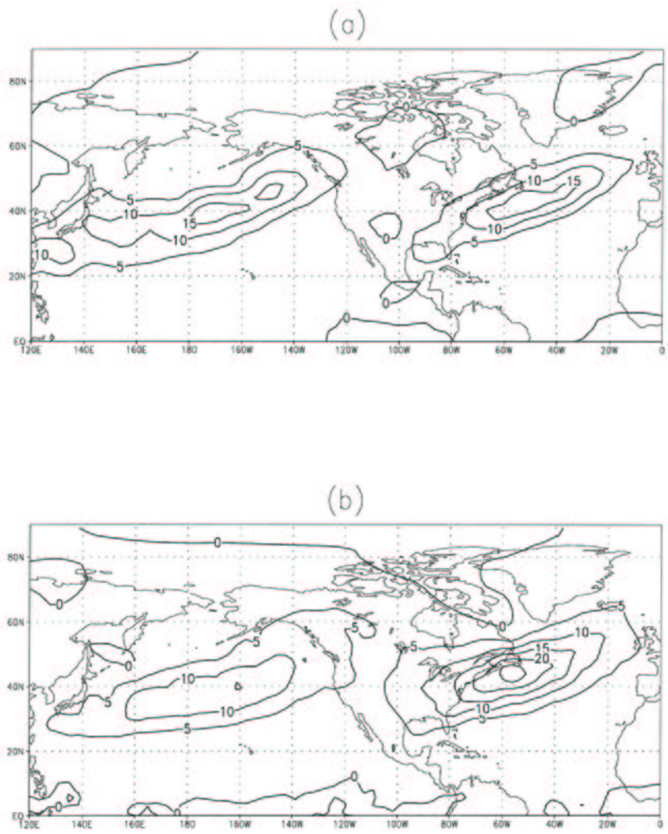

Figure 8: The distribution of transient eddy heat fluxes at $\sigma = 0.865$ *from the linear stochastic model (a) and the R30 perpetual Jan GCM (b). Units:* $ms^{-1}K$*. From Zhang & Held, (1999).*

As $t \to \infty$ and assuming the operator $\mathbf{A}$ is stable, as is the operator about the climatological mean flow, a steady state is reached, which satisfies the steady state Lyapunov equation:

$$\mathbf{A}\mathbf{C}^{\infty} + \mathbf{C}^{\infty}\mathbf{A}^{\dagger} = -\mathbf{Q} \ . \tag{26}$$

This equation can be readily solved for $\mathbf{C}^{\infty}$ from which all the ensemble mean quadratic flux quantities can be derived.

The interpretation of $\mathbf{C}$ requires care. The asymptotic steady state ensemble average $\mathbf{C}^{\infty}$ is equal to the time averaged covariance and can be identified as the climatological time mean state of a single planet as obtained from averaging over a sufficient long interval. However, the time dependent $\mathbf{C}(t)$ can not be associated with the time mean[2] but rather is an ensemble average. In that case the above development is appropriate for estimation of the propagation of the error covariance in ensemble prediction which will be discussed in the next section. In this section we consider a time independent and stable $\mathbf{A}$ and interpret the steady state $\mathbf{C}^{\infty}$ as the climatological covariance of the perturbations to the climatological mean flow. It has been demonstrated that such a formulation accurately models the midlatitude climatology (Farrell & Ioannou, 1994, 1995; DelSole, 1996, 1999, 2001; Whitaker & Sardeshmukh, 1998; Zhang & Held, 1999) and reproduces the climatological heat and momentum fluxes. In Fig. 7 we can see that the asymptotic covariance captures the distribution of the geopotential height covariance of the midlatitude atmosphere and in Fig. 8 and Fig. 9 that it also captures the distribution of heat and momentum flux in the extratropics.

The algebraic equation (26) gives the climatological mean, $\mathbf{C}^{\infty}$, as an explicit functional of the forcing covariance $\mathbf{Q}$ and the functions entering in the climatological mean operator $\mathbf{A}$, such as the structure of the mean

[2]Under certain conditions it can be associated with a zonal mean, for discussion and physical application of this interpretation see Farrell & Ioannou (2002a).

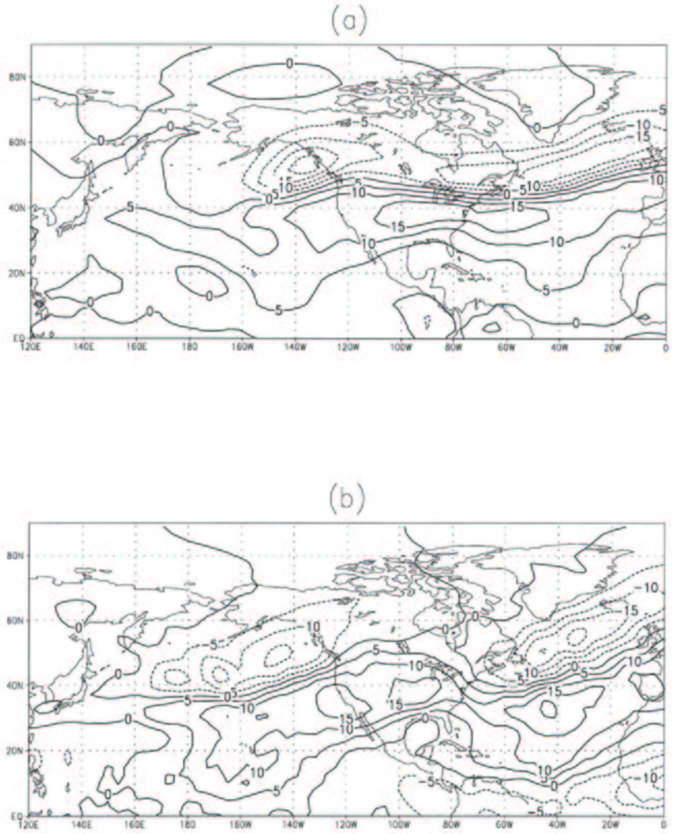

Figure 9: The distribution of transient eddy momentum fluxes at $\sigma = 0.26$ for the linear stochastic model (a) and the R30 perpetual Jan GCM (b). Units:m^2s^{-2}. From Zhang & Held, (1999).

flow and the structure of the dissipation parameters. This formulation permits a systematic investigation of the sensitivity of the climate to changes in the forcing and distribution of the mean flow.

We address the sensitivity of the climate to changes in the forcing under the assumption that the climatological mean flow is fixed.

We want to determine the forcing structure given by the column of $\mathbf{F}$ that contributes most to the ensemble average variance $< E(t) >$. This structure will be called stochastic optimal (Farrell & Ioannou, 1996; Kleeman & Moore, 1997).

The ensemble average variance can be shown to be:

$$< E(t) >=< \psi^\dagger \psi > \ = \mathbf{F}^\dagger \mathbf{B}(t) \mathbf{F} \ , \tag{27}$$

where $\mathbf{B}(t)$ is the stochastic optimal matrix:

$$\mathbf{B}(t) \ = \ \int_0^t e^{\mathbf{A}^\dagger s} e^{\mathbf{A}s} \, ds \ . \tag{28}$$

The stochastic optimal matrix satisfies the time dependent back Lyapunov equation, analogous to (24):

$$\frac{d\mathbf{B}}{dt} \ = \mathbf{BA} \ + \ \mathbf{A}^\dagger \mathbf{B} \ + \ \mathbf{I} \ . \tag{29}$$

If $\mathbf{A}$ is stable the statistical steady state $\mathbf{B}^\infty$ satisfies the algebraic equation:

$$\mathbf{B}^\infty \mathbf{A} \ + \ \mathbf{A}^\dagger \mathbf{B}^\infty \ = - \ \mathbf{I} \ , \tag{30}$$

which can be readily solved for $\mathbf{B}^\infty$.

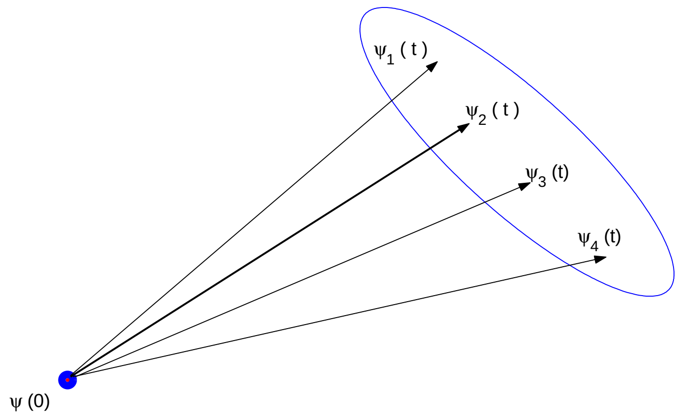

Figure 10: Schematic evolution of a sure initial condition $\psi(0)$ in an uncertain system. After time t the evolved states $\psi(t)$ lie in the region shown. Initially the covariance matrix $\mathbf{C}(0) = \psi(0)\psi^\dagger(0)$ is rank 1, but at time t the covariance matrix is of rank greater than 1. For example if the final states were $\psi_i(t)$ $(i = 1, \cdots, 4)$ with equal probability, the covariance at time t: $\mathbf{C}(t) = \frac{1}{4}\sum_{i=1}^{4} \psi_i(t)\psi_i^\dagger(t)$ would be rank 4, representing an entangled state. In contrast, in certain systems the degree of entanglement is invariant and a pure state evolves to a pure state.

Having obtained $\mathbf{B}^\infty$ from (27) we obtain the stochastic optimal as the eigenfunction of $\mathbf{B}^\infty$ associated with the largest eigenvalue. The stochastic optimal determines the forcing structure that is most effective in producing variance. General forcings will have greatest impact on the maintained variance when they predominantly project on the top stochastic optimals (the top eigenfunctions of $\mathbf{B}^\infty$).

3 Statistical Predictability of Uncertain Systems

3.1 The case of additive uncertainty

Consider a tangent linear system with additive model error. On the assumption that the model error can be treated as a stochastic forcing to the tangent linear system the errors evolve according to:

$$\frac{d\psi}{dt} = \mathbf{A}(t)\psi + \mathbf{F}\eta(t) \ , \tag{31}$$

where $\mathbf{A}(t)$ is the tangent linear operator which is considered certain. Such systems are uncertain and as a result a single initial state may produce a variety of states at a time later depending on the realization of the stochastic process η. This is shown schematically in Fig. 10. The covariance here is interpreted as the mean error covariance in the tangent linear model.

Let us assume that initially the model error covariance is $\mathbf{C}(0)$. At a time t later the error covariance is given by (25). The homogeneous solution of (25) represents the covariance that results from the deterministic evolution of $\mathbf{C}(0)$ and represents error growth associated with uncertainty in the specification of the initial state of the atmosphere. Predictability studies traditionally concentrate on this source of error growth. The inhomogeneous part of (25) represents the contribution of model error.

The deterministic (free) growth of the error covariance at any time is bounded by the growth associated with the optimal perturbation. The forced error growth at any time on the other hand is bounded by the error covariance at time t forced by the different stochastic optimal that is determined from the stochastic optimal matrix $\mathbf{B}(t)$ at the same time t. Given an initial error covariance and a forcing covariance it is of interest to determine the time at which the accumulated covariance from the model error exceeds the error produced from the uncertainty in the initial conditions. As an example consider the simple system (7). Let us assume that initially the state has error such that $\text{trace}(\mathbf{C}(0)) = 1$ and that the model error has covariance $\text{trace}(\mathbf{Q}) = 1$. The growth of errors due

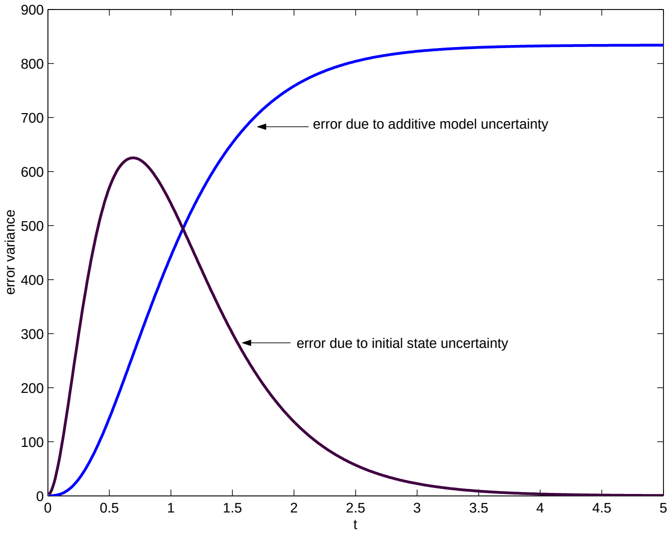

Figure 11: Evolution of error variance associated with free evolution of the optimal covariance with unit initial variance and evolution of error variance forced by additive uncertainty with unit forcing variance. The operator **A** *is the simple* 2×2 *Reynolds operator given in (7).*

to uncertainty in the initial conditions is plotted as a function of time in Fig. 11. After approximately unit time the error covariance is dominated by the accumulated error from model uncertainty.

This simple example shows that as the initial state is more accurately determined error growth will inevitably be dominated by model error. At present the success of the deterministic forecasts and increase in forecast accuracy obtained by decreasing initial state error suggest that improvements in forecast accuracy are still being achieved by reducing the uncertainty in the initial state.

3.2 The case of multiplicative uncertainty

Consider now a forecast system with uncertain parameterizations so that the tangent linear system operator is itself uncertain and for simplicity takes the form:

$$\mathbf{A}(t) = \mathbf{A} + \varepsilon \mathbf{B} \eta(t) \ , \tag{32}$$

where $\eta(t)$ a stochastic process with zero mean and unit variance, $\mathbf{B}$ a fixed matrix characterizing the uncertainty, and ε an amplitude factor.

Fix the inner product with which the perturbation magnitude is measured and concentrate on the calculation of the error growth. Because of the uncertainty in the operator different realizations of η will result in different perturbation amplitudes and given that the probability density function of η is known, each amplitude can be ascribed a probability. A measure of the error growth is the expectation of the various growths:

$$< g >= \int P(\omega) g(\omega) d\omega \ , \tag{33}$$

where ω is a realization of η, $P(\omega)$ is the probability for its occurrence, and $g(\omega)$ the error growth associated with this realization of the operator. Because of the convexity of the expectation the r.m.s. second moment error growth exceeds the amplitude error growth, i.e.

$$\sqrt{< g^2 >} \geq < g > \quad . \tag{34}$$

It follows that in uncertain systems different moments generally give different growth estimates and the possibility arises that the Lyapunov exponent of an uncertain system may be negative while higher moments are unstable. This emphasizes the fact that rare or extreme events that are weighted more by the higher moment measure are more difficult to predict.

As an example consider two classes of trajectories which with equal probability give growth in unit time of $g = 2$ or $g = 1/2$. What is the expected growth in unit time? While the Lyapunov growth rate is 0 because:

$$\lambda \ = \ \frac{< \log\ g >}{t} \ = \ \frac{\log 2}{2} + \frac{\log(1/2)}{2} \ = \ 0\ , \tag{35}$$

the second moment growth rate is positive:

$$\lambda_2 \ = \ \frac{\log < g^2 >}{t} \ = \ \log\left(\frac{1}{2}\times 4+\frac{1}{2}\times\frac{1}{4}\right) \ = \ 0.75\ , \tag{36}$$

proving that for this simple example the error covariance increases exponentially fast and showing that uncertain systems may be Lyapunov (sample) stable, while higher order moments are unstable. In fact if the uncertainty is Gaussian there is always a higher moment that grows exponentially (cf. Farrell & Ioannou, 2002b).

The implication of this property of uncertain systems is that the associated optimal error growth is not derivable from the norm of the ensemble mean propagator. To obtain the optimal growth we must determine the evolution of the covariance $\mathbf{C} =< \psi\psi^\dagger >$ under uncertain dynamics, then determine the optimal $\mathbf{C}(0)$ of unit trace that leads to greatest trace$(\mathbf{C}(\mathrm{t}))$ at later times.

3.3 The Optimal in an Uncertain System

Consider an uncertain tangent linear system of the form:

$$\frac{d\psi}{dt} \ = \ \mathbf{A}\psi \ + \ \varepsilon\eta(t)\mathbf{B}\psi\ , \tag{37}$$

where $\mathbf{A}$ is the sure mean operator and $\mathbf{B}$ is the structure of the uncertainty in the operator and $\eta(t)$ is its time dependence. Take $\eta(t)$ to be a Gaussian random variable with zero mean, unit variance and autocorrelation time t_c. Define $\Phi(t,0)$ to be the propagator for a realization of the operator $\mathbf{A}+\varepsilon\eta(t)\mathbf{B}$. For that realization the variance at time t is:

$$\psi(t)^\dagger\psi(t) \ = \ \psi(0)^\dagger\Phi^\dagger(t,0)\Phi(t,0)\,\psi(0) \tag{38}$$

where $\psi(0)$ is the initial state. The optimal initial state, i.e. the initial state that leads to the greatest variance at time t, for this realization is the eigenvector of

$$\mathbf{H}(t) = \Phi^\dagger(t,0)\Phi(t,0) \tag{39}$$

with largest eigenvalue.

For uncertain dynamics we seek the greatest expected variance at t by determining the ensemble average

$$< \mathbf{H}(t) >= \left\langle \Phi^\dagger(t,0)\Phi(t,0)\right\rangle \ . \tag{40}$$

The optimal initial condition is identified as the eigenvector of $< \mathbf{H}(t) >$ with largest eigenvalue. This eigenvalue determines the optimal error growth. This gives constructive proof of the remarkable fact that there is a single sure initial condition that maximizes expected error growth in an uncertain tangent linear system.

It can be shown (cf Farrell & Ioannou, 2002c,d) that for Gaussian fluctuations $< \mathbf{H}(t) >$ evolves according to the exact equation:

$$\frac{d < \mathbf{H} >}{dt} \ = \ \left(\mathbf{A}+\varepsilon^2\mathbf{E}(t)\mathbf{B}\right)^\dagger < \mathbf{H}(t) > \ + \ < \mathbf{H}(t) > \left(\mathbf{A}+\varepsilon^2\mathbf{E}(t)\mathbf{B}\right) \ + \tag{41}$$

$$+ \ \ \varepsilon^2\left(\mathbf{E}^\dagger(t) < \mathbf{H} > \mathbf{B} \ + \ \mathbf{B}^\dagger < \mathbf{H} > \mathbf{E}(t)\right) \tag{42}$$

where

$$\mathbf{E}(t) \;=\; \int_0^t e^{-\mathbf{A}s}\mathbf{B}e^{\mathbf{A}s}e^{-\nu s}\,ds \;\; . \tag{43}$$

For small autocorrelation times the above equation reduces to:

$$\frac{d < \mathbf{H} >}{dt} = \left(\mathbf{A} + \frac{\varepsilon^2}{\nu}\mathbf{B}^2\right)^{\dagger} < \mathbf{H}(t) > + \; < \mathbf{H}(t) > \left(\mathbf{A} + \frac{\varepsilon^2}{\nu}\mathbf{B}^2\right) + \frac{2\varepsilon^2}{\nu}\mathbf{B}^{\dagger} < \mathbf{H} > \mathbf{B} \;\; . \tag{44}$$

As an example application of this result, the ensemble for an uncertain tangent linear system should be constructed from this basis of optimals, i.e. the optimals of $< \mathbf{H} >$.

4 Conclusions

We briefly summarize the main points of this review:

1) Generalized stability theory of deterministic operators (autonomous or non-autonomous) concentrates attention on the optimal perturbations. These are obtained by singular value analysis of the propagator or equivalently by repeated forward integration of the system followed by backward integration of the adjoint system.

2) The optimal perturbations are used to understand and predict error growth and structure.

3) In time dependent systems the unstable error lies in the subspace of the leading optimal (singular) vectors.

4) Generalized stability theory allows us to address questions of predictability of statistics and of sensitivity of statistics to changes in the forcing and changes in the operator.

5) Generalized stability theory addresses expected error growth in the presence of model and trajectory uncertainties.

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