Non-hydrostatic modelling with the COSMO model

ECMWF Workshop on non-hydrostatic modelling
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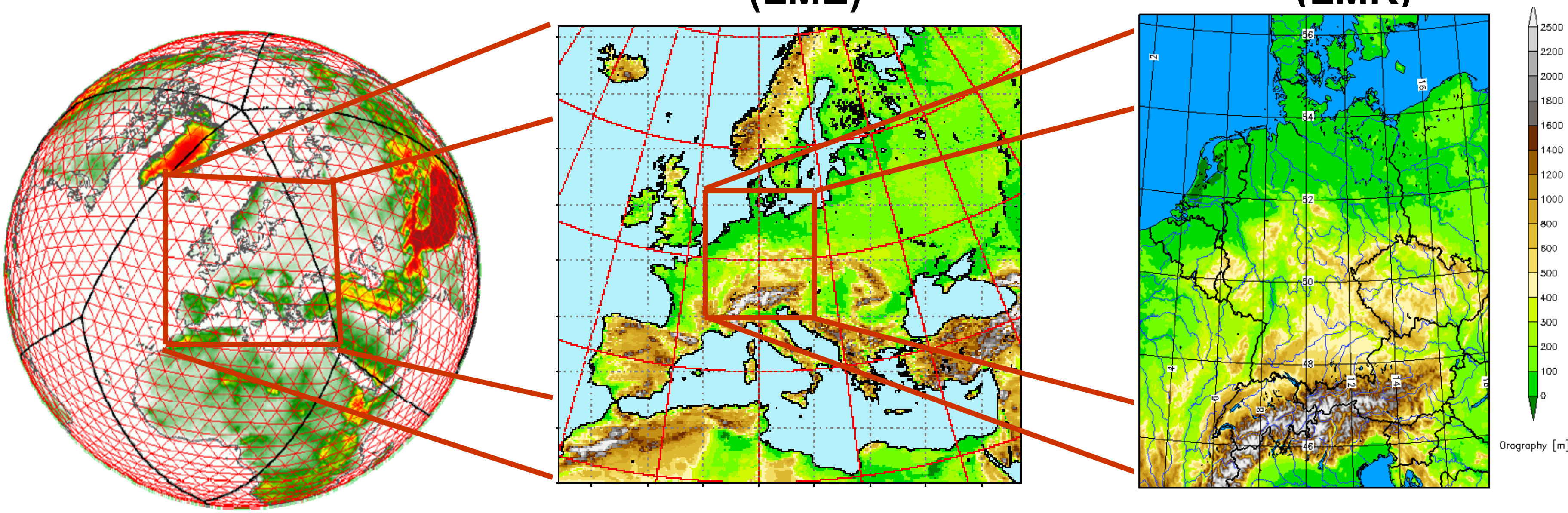
Outlook

- model chain at DWD
- stability analysis of the Runge-Kutta (RK) scheme
- introduction of Runge-Kutta (RK) scheme in COSMO-EU
- outlook to current/future dynamical core developments for COSMO
    - wave expansion properties of the anelastic / compressible + divergence damping equations

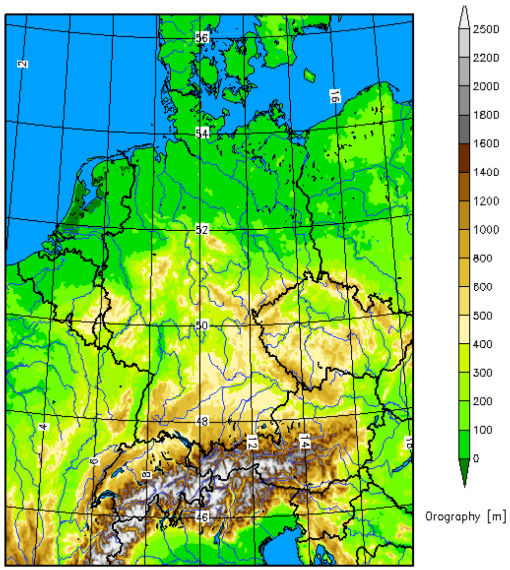
The operational Model Chain of DWD: GME, COSMO-EU and -DE

GME	COSMO-EU (LME)	COSMO-DE (LMK)



GME	COSMO-EU (LME)	COSMO-DE (LMK)
hydrostatic	non-hydrostatic	non-hydrostatic
parameterised convection	parameterised convection	convection-permitting
$\Delta x \approx 30$ km	$\Delta x = 7$ km	$\Delta x = 2.8$ km
655362 * 60 GP	665 * 657 * 40 GP	421 * 461 * 50 GP
$\Delta t = 100$ sec., T = 7 days	$\Delta t = 66$ sec., T = 78 h	$\Delta t = 25$ sec., T = 21 h
		(since 16. April 2007)

COSMO-DE
(LMK)



Baldauf, Seifert, Majewski, …, subm. to MWR

Basic time integration scheme for dynamical core:
3-stage Runge-Kutta
(*Wicker, Skamarock, 2002, MWR*)

Efficiency achieved by:

- time-splitting:
  slow process with large time step ΔT = 25 sec.
  fast process with small time step Δt = 4.16 sec
- Implicit vertical discretization

Questions:

- Which stability statements can be made?
- Stability analysis in time <u>and</u> space
- Temporal order of the integration scheme?
- possible improvements/alternatives?

The current 3-stage RK-scheme in the COSMO-model

Wicker, Skamarock (2002) MWR

Vertical advection — **Horizontal advection**

Coriolis force, physics tendencies,

pressure gradient, divergence terms, …

$$\text{solve the implicit scheme:} \quad \frac{\phi - \phi^n}{\frac{\Delta t}{3}} = \beta A_z(\tilde{\phi}) + (1-\beta) A_z(\phi^n) + A_x(\phi^n) + P(\phi^n)$$

$$\text{... and define its tendency:} \quad L(\phi^n) := \frac{\tilde{\phi} - \phi^n}{\frac{\Delta t}{3}}$$

$$\text{1. RK-substep:} \quad \phi^* = \phi^n + \frac{\Delta t}{3} L(\phi^n)$$

$$\text{fast waves with tendency} \quad \frac{\phi^* - \phi^n}{\Delta t/3}, \text{ starting at } \phi^n \Rightarrow \phi^*$$

$$\text{solve:} \quad \frac{\tilde{\phi} - [\alpha \phi^n + (1-\alpha)\phi^*]}{\frac{\Delta t}{2}} = \beta A_z(\tilde{\phi}) + (1-\beta) A_z(\phi^*) + A_x(\phi^*) + P(\phi^n)$$

$$\text{... and define its tendency:} \quad L(\phi^*) := \text{lhs. of the above expression}^1$$

$$\text{2. RK-substep:} \quad \phi^{**} = \phi^n + \frac{\Delta t}{2} L(\phi^*)$$

$$\text{fast waves with tendency} \quad \frac{\phi^{**} - \phi^n}{\Delta t/2}, \text{ starting at } \phi^n \Rightarrow \phi^{**}$$

$$\text{solve:} \quad \frac{\tilde{\phi} - [\alpha \phi^n + (1-\alpha)\phi^{**}]}{\Delta t} = \beta A_z(\tilde{\phi}) + (1-\beta) A_z(\phi^{**}) + A_x(\phi^{**}) + P(\phi^n)$$

$$\text{... and define its tendency:} \quad L(\phi^{**}) := \text{lhs. of the above expression}$$

$$\text{3. RK-substep:} \quad \phi^{n+1} = \phi^n + \Delta t \, L(\phi^{**})$$

$$\text{fast waves with tendency} \quad \frac{\phi^{n+1} - \phi^n}{\Delta t}, \text{ starting at } \phi^n \Rightarrow \phi^{n+1}$$

In the following stability analysis: no vertical advection considered ←→ α=0

Temporal order conditions for the time-split RK3

Deutscher Wetterdienst
Wetter und Klima aus einer Hand

slow processes (e.g. advection):

$$Q_s = 1 + \Delta T\, P_s^{(1)} + \frac{\Delta T^2}{2!} P_s^{(2)} + \frac{\Delta T^3}{3!} P_s^{(3)} + O(\Delta T^4)$$

fast processes (e.g. sound, gravity wave expansion):

$$Q_f = 1 + \Delta t\, P_f^{(1)} + \frac{\Delta t^2}{2!} P_f^{(2)} + \frac{\Delta t^3}{3!} P_f^{(3)} + O(\Delta t^4)$$

Insertion into the time-split RK3WS integration scheme: $n_s := \Delta T / \Delta t$

$$Q_{RK3WS} = 1 + \Delta T \left[P_s^{(1)} + P_f^{(1)} \right] + \frac{\Delta T^2}{2} \left[P_s^{(2)} + (P_s^{(1)})^2 + P_s^{(1)} P_f^{(1)} + \right.$$
$$\left. + \left(1 - \frac{1}{n_s}\right) P_f^{(1)} P_s^{(1)} + \left(1 - \frac{1}{n_s}\right) (P_f^{(1)})^2 + \frac{1}{n_s} P_f^{(2)} \right] + O(\Delta T)^3$$

compare with exp [$\Delta T\,(P_s + P_f)$] → at most 2nd order, only for:
slow process = Euler forward and $n_s \to \infty$; fast process can be of 2nd order

Linear advection equation (1-dim.)

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0$$

discretized:

$$\frac{q_j^{n+1} - q_j^n}{\Delta t} = f_j(q^n)$$

Spatial discretisations of the advection operator (order 1 ... 6) (u>0 assumed)

$f_j^{(1)}(q) := -u\frac{q_j - q_{j-1}}{\Delta x}$	upwind 1st order	`iadv_order=1`
$f_j^{(2)}(q) := -u\frac{q_{j+1} - q_{j-1}}{2\,\Delta x}$	centered diff. 2nd order	`iadv_order=2`
$f_j^{(3)}(q) := -u\frac{2q_{j+1} + 3q_j - 6q_{j-1} + q_{j-2}}{6\,\Delta x}$		`iadv_order=3`
$f_j^{(4)}(q) := -u\frac{-(q_{j+2} - q_{j-2}) + 8(q_{j+1} - q_{j-1})}{12\,\Delta x}$		`iadv_order=4`
$f_j^{(5)}(q) := -u\frac{-3q_{j+2} + 30q_{j+1} + 20q_j - 60q_{j-1} + 15q_{j-2} - 2q_{j-3}}{60\,\Delta x}$		**`iadv_order=5`**
$f_j^{(6)}(q) := -u\frac{(q_{j+3} - q_{j-3}) - 9(q_{j+2} - q_{j-2}) + 45(q_{j+1} - q_{j-1})}{60\,\Delta x}$		`iadv_order=6`

from the Theorem and Lemma below:
all Nth order (LC-) Runge-Kutta-schemes have the same linear stability properties for u=const. !

Hundsdorfer et al. (1995) JCP
Wicker, Skamarock (2002) MWR

Stable Courant-numbers for advection schemes

	up1	cd2	up3	cd4	up5	cd6
LC-RK1 (Euler)	1	0	0	0	0	0
LC-RK2	1	0	0.874	0	0	0
LC-RK3	1.256	1.732	1.626	1.262	1.435	1.092
LC-RK4	1.393	2.828	1.745	2.061	1.732	1.783
LC-RK5	1.609	0	1.953	0	1.644	0
LC-RK6	1.777	0	2.310	0	1.867	0
LC-RK7	1.977	1.764	2.586	1.286	2.261	1.113

Stability limit for the ‚effective Courant-number' for advection schemes

$C_{\text{eff}} := C \,/\, s$, s = stage of RK-scheme

	up1	cd2	up3	cd4	up5	cd6
Euler	**1**	0	0	0	0	0
LC-RK2	0.5	0	0.437	0	0	0
LC-RK3	0.419	**0.577**	**0.542**	0.421	**0.478**	0.364
LC-RK4	0.348	**0.707**	0.436	**0.515**	0.433	0.446
LC-RK5	0.322	0	0.391	0	0.329	0
LC-RK6	0.296	0	0.385	0	0.311	0
LC-RK7	0.282	0.252	0.369	0.184	0.323	0.159

Baldauf (2008): Stability analysis for linear discretizations of the advection equation with Runge-Kutta time integration, J. Comput. Phys.

→ General theory for Linear Case (LC)-Runge-Kutta-schemes, applied to the Dahlquist test problem

Stability analysis of a 2D (horizontal + vertical) system with Sound + Buoyancy + Advection (+ Smoothing, Filtering)

Advection — Sound — 'buoyancy'

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + Q_x \\
\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\left(\frac{T'}{T_0} - \frac{p'}{p_0}\right) + Q_z \\
\frac{\partial p'}{\partial t} + U_0 \frac{\partial p'}{\partial x} &= -\frac{c_p}{c_v} p_0 D + \rho_0 g w \\
\frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} &= -\frac{R}{c_v} T_0 D - \frac{\partial T_0}{\partial z} w \\
D &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}
\end{aligned}
$$

$p = p_0 + p'$

$T = T_0 + T'$

Divergence damping:

$$Q_x = \alpha_{div} \frac{\partial D}{\partial x}$$

$$Q_z = \alpha_{div} \frac{\partial D}{\partial z}$$

Von Neumann stability analysis → Restrictions:

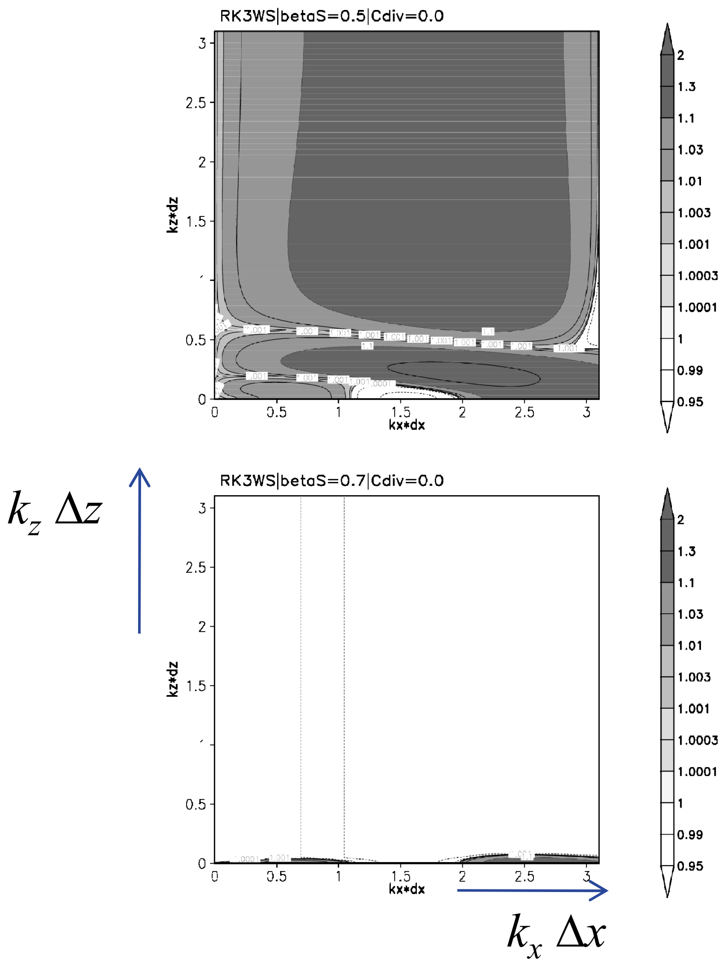
- no boundaries (wave expansion in ∞ extended medium)
- base state: p_0=const, T_0=const (for the coefficients)
  (but stratification $dT_0/dz \neq 0$ possible)
  → application to structures with a vertical extension << 10 km
- no orography (i.e. no metric terms)
- only horizontal advection

Stability of single waves for advection + sound
(partial operator splitting with RK3, C_{adv}=1.0, $C_{snd,x}$=0.6)

(without divergence damping)

β^s=0.5

β^s=0.7



Sound term discretization:

Spatial: centered differences

Temporal:
Forward-backward (*Mesinger, 1977*)
Vertically implicit ($\beta^s \geq$ ½)

→stable for $C_{snd,x}$ < 1, $C_{snd,z}$ arbitrary

Divergence damping

$$\frac{\partial \mathbf{v}}{\partial t} = \ldots + \alpha_{div} \nabla \, \mathrm{div} \, \mathbf{v}$$

leads to a diffusion of divergence → damping of sound waves
Isotropic div. damping recommended: *Gassmann, Herzog (2007) MWR*

Courant-numbers: $C_{div,x} := \alpha_{div} \frac{\Delta t}{\Delta x^2}$ $C_{div,z} := \alpha_{div} \frac{\Delta t}{\Delta z^2}$

2D, explicit, stagg. grid: stable for $C_{div,x} + C_{div,z} < ½$
2D, hor. expl.-vert. impl., stagg. grid: stable for $C_{div,x} < ½$, $C_{div,z}$ arbitrary

$C_{div,x}$ = 0.1 für COSMO-DE → α_{div}=160000 m²/s → $\alpha_{div} / (c_s^2 \Delta t) \sim 0.3$

Skamarock, Klemp (1992) MWR, Wicker, Skamarock (2002) MWR

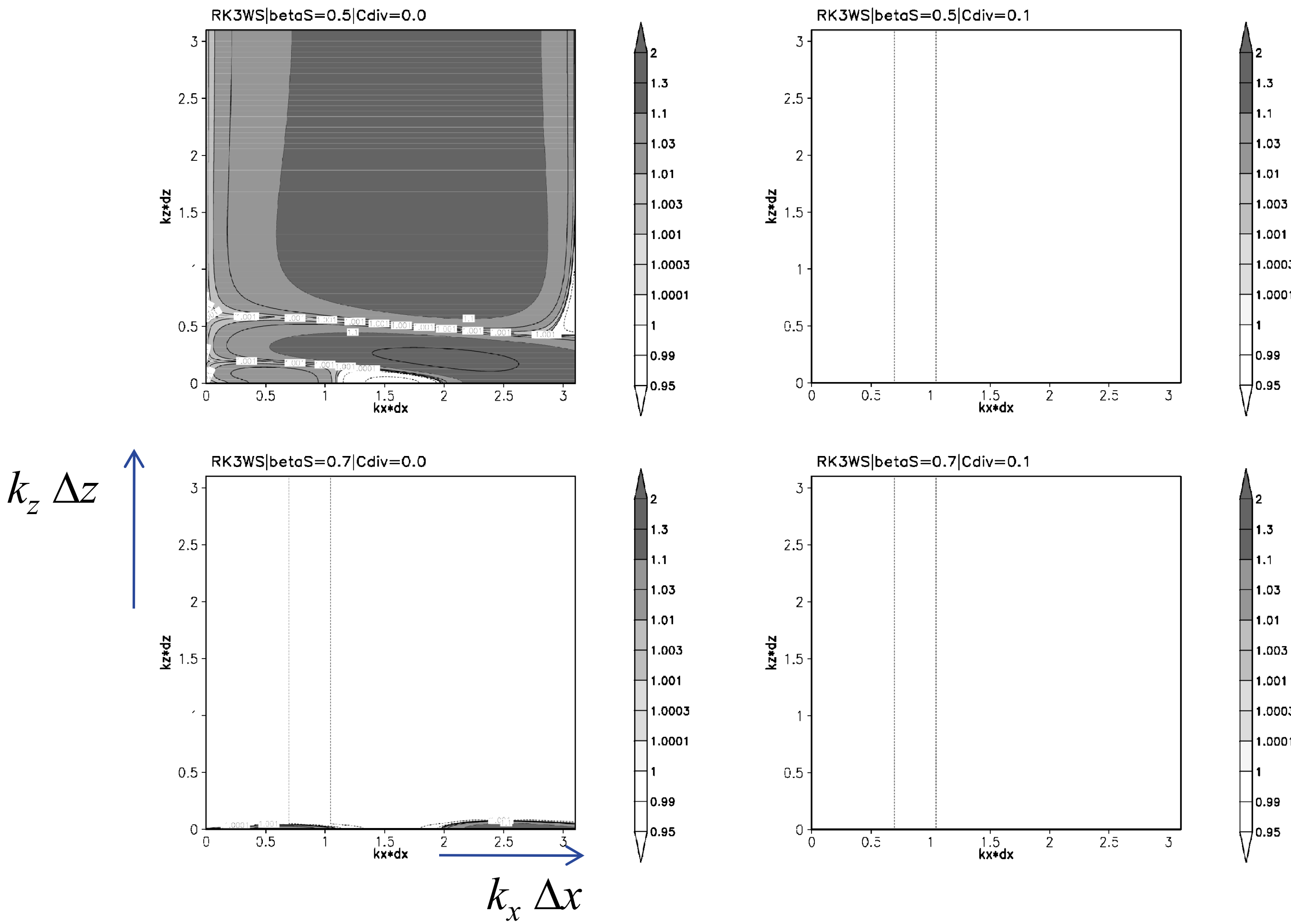
Stability of single waves for advection + sound

(partial operator splitting with RK3, C_{adv}=1.0, $C_{snd,x}$=0.6)

without div.-damping | with div.-damping

β^s=0.5

β^s=0.7



Buoyancy terms:

$$
\begin{aligned}
\frac{w^{n+1} - w^n}{\Delta t} &= g \left(\beta_T^b \frac{T'^{n+1}}{T_0} + (1-\beta_T^b)\frac{T'^n}{T_0} - \beta_p^b \frac{p'^{n+1}}{p_0} - (1-\beta_p^b)\frac{p'^n}{p_0} \right) \\
\frac{p'^{n+1} - p'^n}{\Delta t} &= \rho_0 g (\beta_3^b\, w^{n+1} + (1-\beta_3^b)\, w^n) \\
\frac{T'^{n+1} - T'^n}{\Delta t} &= -\frac{\partial T_0}{\partial z} (\beta_4^b\, w^{n+1} + (1-\beta_4^b)\, w^n)
\end{aligned}
$$

acoustic cut-off frequency $\omega_a := \sqrt{N^2 + \frac{g^2}{c_S^2}}$; $C_{buoy} := \omega_a \Delta t$

$C_\beta = \frac{1}{T_0} \frac{\partial T_0}{\partial z} \frac{c_S^2}{g} \approx -0.24$ (standard atmosphere)

fully explicit	unstable	-
forward-backward	stable for $C_{buoy} < 2$	neutral
Crank-Nicholson $\beta = 1/2$	uncond. stable	neutral
Crank-Nicholson $\beta > 1/2$	uncond. stable	damping
implicit	uncond. stable	damping

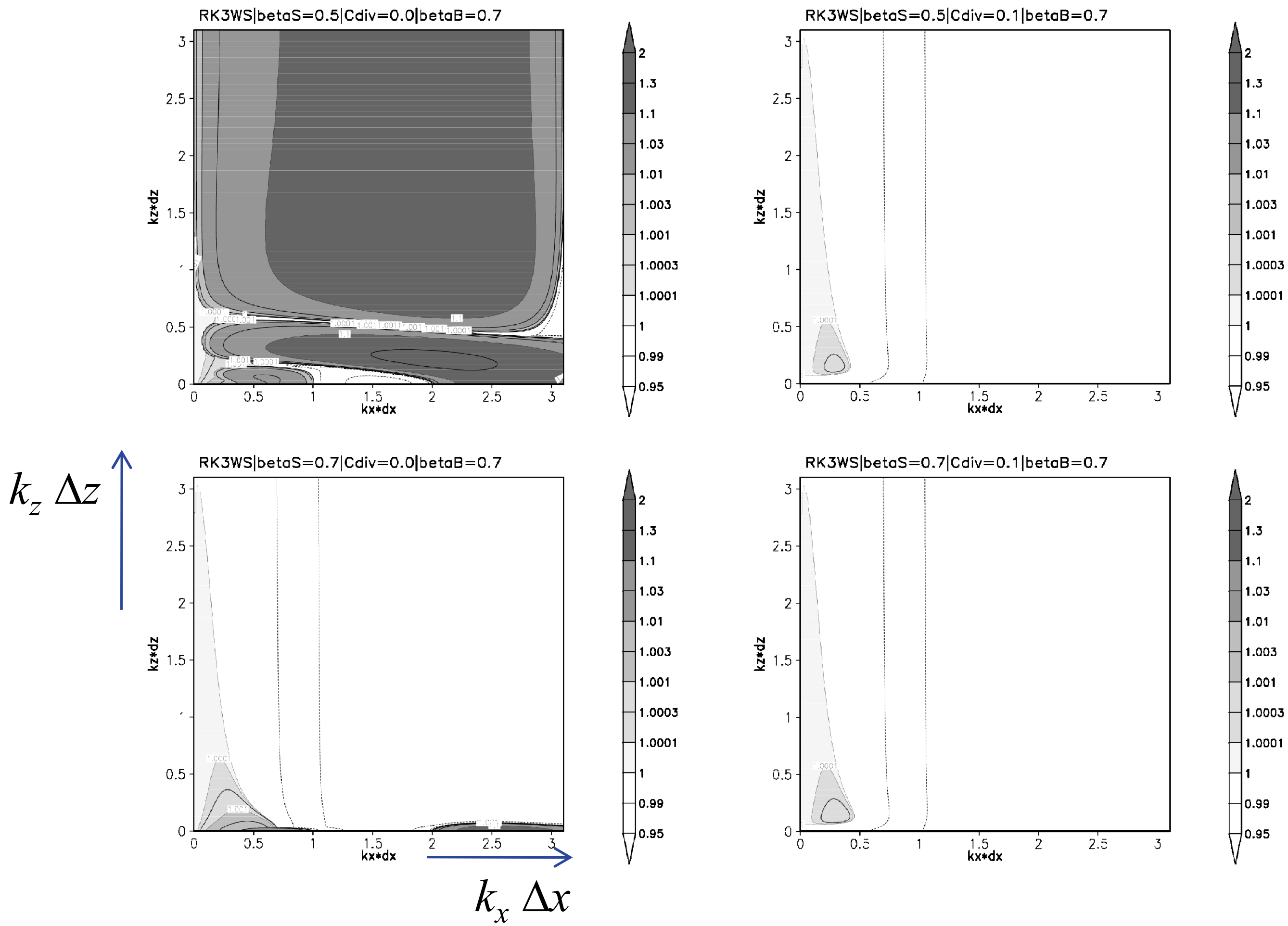
Stability of single waves for advection + sound + buoyancy

(partial operator splitting with RK3, C_{adv}=1.0, $C_{snd,x}$=0.6)

without div.-damping | with div.-damping

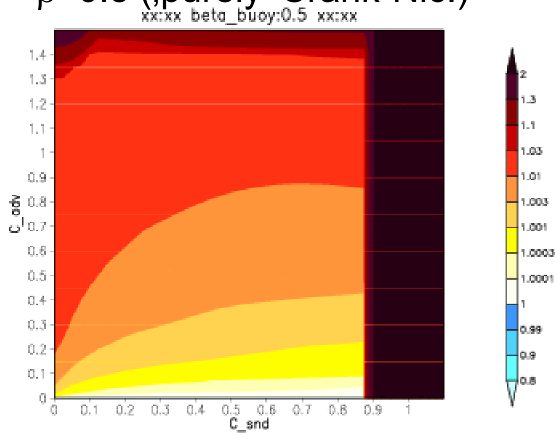
β^s=0.5

β^s=0.7

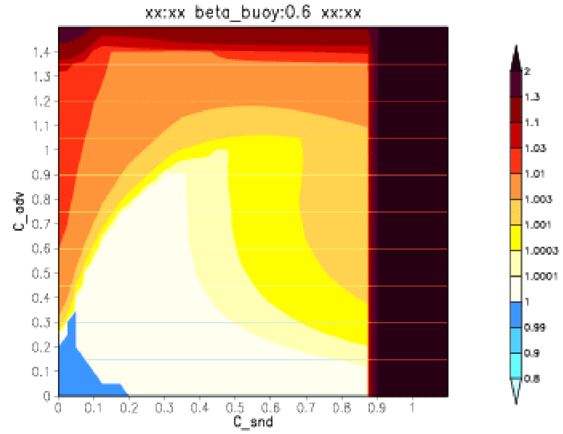


Choice of CN-parameters for buoyancy in the *p‘T‘*-Dynamic of COSMO-DE

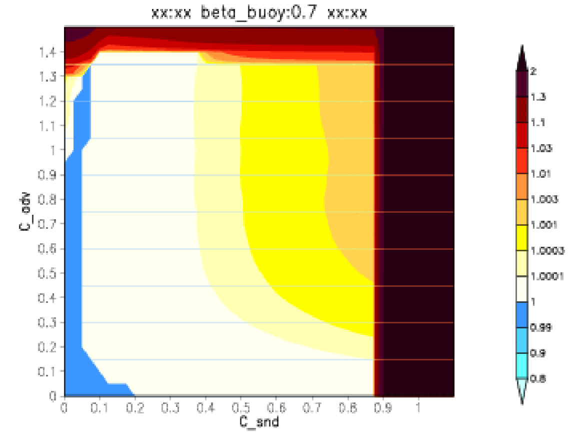
β=0.5 (‚purely‘ Crank-Nic.)



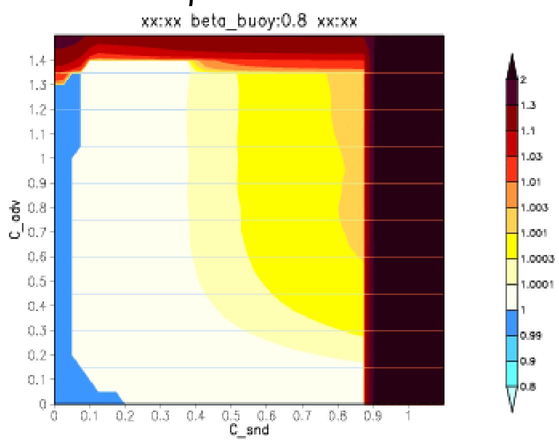
β=0.6



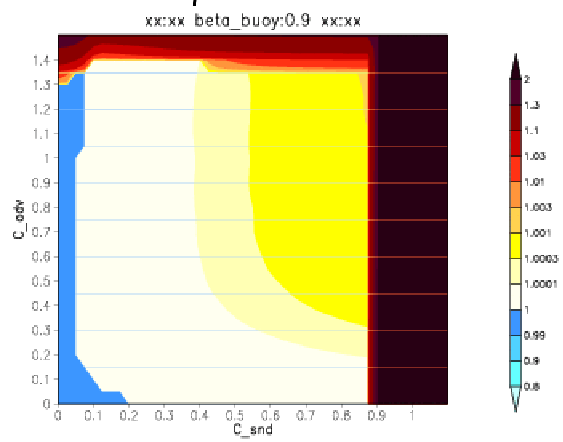
β=0.7



β=0.8

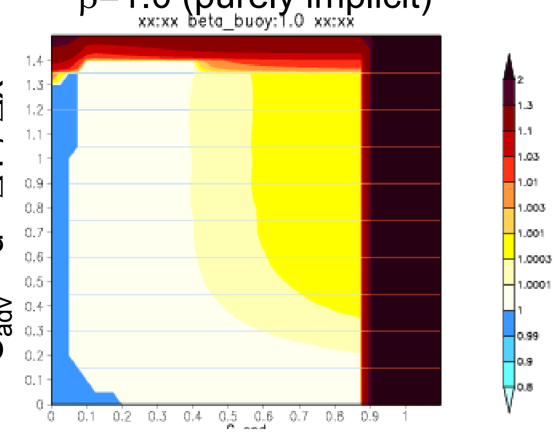


β=0.9



β=1.0 (purely implicit)

$C_{adv} = u * \Delta T / \Delta x$



$C_{snd} = c_s * \Delta t / \Delta x$

→ **choose β=0.7 as the best value**

'Simplest'-LC-RK4
(4-stage, 2nd order)

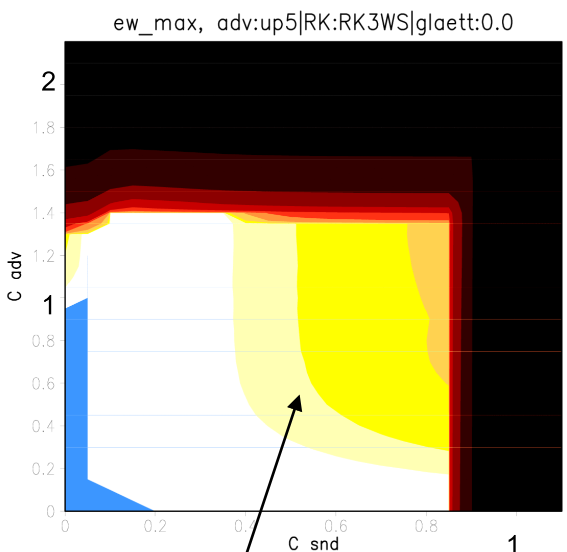
0				
¼	¼			
1/3	0	1/3		
½	0	0	½	
	0	0	0	1

Are 4-stage RK-schemes competitive?

S-LC-RK3
(3-stage, 2nd order) **+ up5**

$\Delta T/\Delta t = 6$

no smoothing



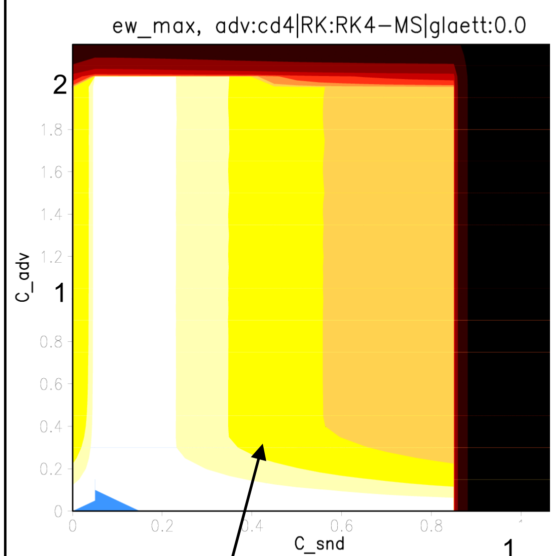
instab. by long waves

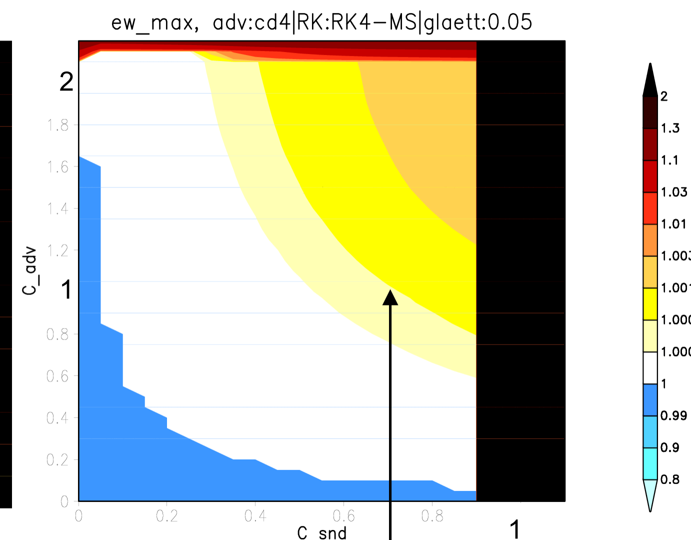
S-LC-RK4 (4-stage, 2nd order) **+ cd4**

$\Delta T/\Delta t = 12$

no smoothing

+ 4th order diffusion





instab. by short waves

instab. by long waves

Are 4-stage RK-schemes competitive?

'Simplest'-LC-RK4
(4-stage, 2nd order)

0				
¼	¼			
1/3	0	1/3		
½	0	0	½	
	0	0	0	1

'classical' RK4
(4-stage, 4th order)
(Numerical recipies)

0				
½	½			
½	0	½		
1	0	0	1	
	1/6	1/3	1/3	1/6

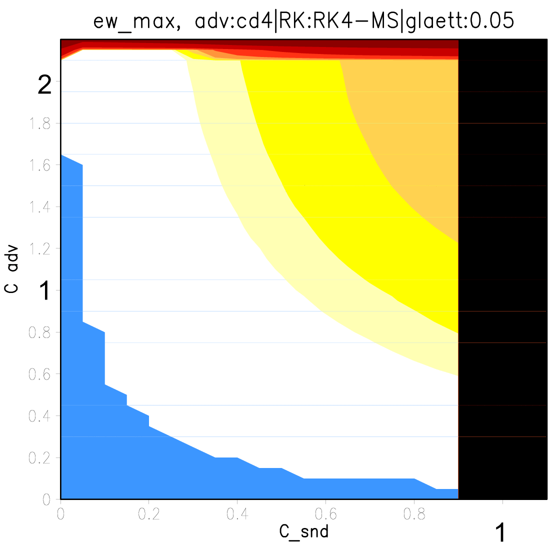
(negative coeff. arises if written in 'substepping-form')

RK-SSP(4,3)
(4-stage, 3rd order)
(Ruuth, Spiteri, 2004)

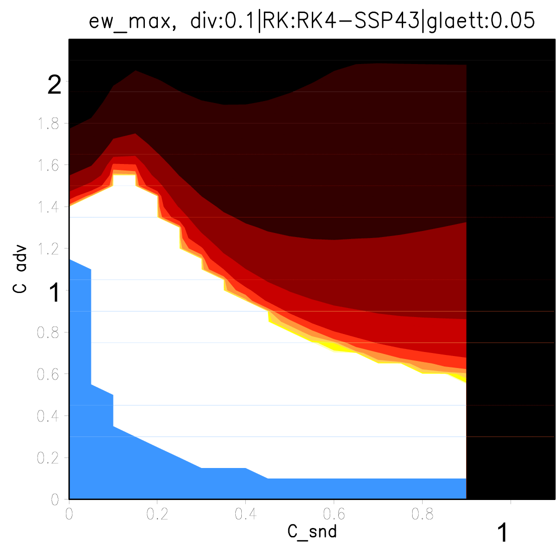
0				
1/2	1/2			
1	1/2	1/2		
1/2	1/6	1/6	1/6	
	1/6	1/6	1/6	1/2

Other RK4 + cd4 + add. smoothing ...

S-LC-RK4
(4-stage, 2nd order)

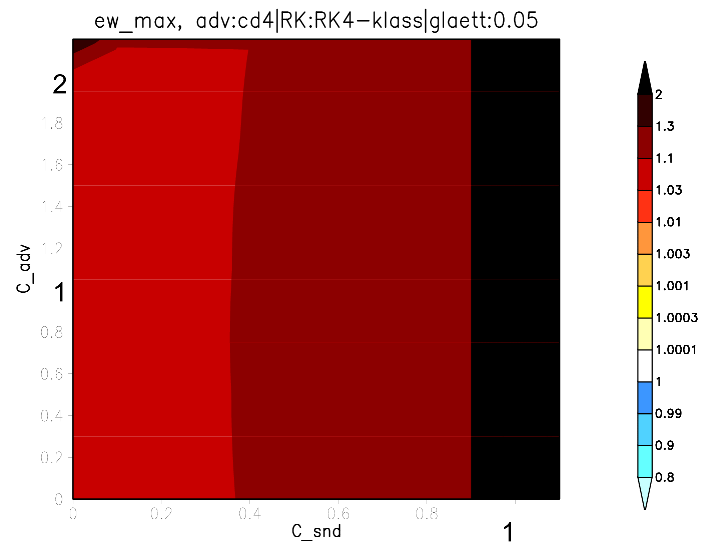


SSP(4,3)-RK
4-stage, 3rd order
(Ruuth, Spiteri, 2004)
<u>is not a LC-RK-scheme!</u>



strong stability preserving schemes do not automatically work together with timesplitting

'classical' RK4
(4-stage, 4th order)



completely unstable (!?) due to negative coefficients in the ‚substepping-form'?

Summary

- RK3 time-splitting: at most 2nd order in time, only for Euler-forward advection and $n_s \to \infty$; fast processes (sound, gravity waves) of 2nd order helps for finite n_s
- von Neumann stability-analysis of 2D Euler equations (Sound-Buoyancy-Advection) in time and space:
  - without metric terms: no off-centering for sound needed (β^s = ½) (with metric: some off-centering recommendable)
  - C_{div} ~ 0.1 is recommended
  - buoyancy needs off-centering (β^B = 0.7, and even then is slightly unstable)
  - overall stability properties are fine for short range NWP and (probably) for regional (limited area) climate modeling
- RK4 + cd4 + weak add. smoothing could be a more efficient alternative to RK3 + up5; with better advection properties than RK3 + up3

Baldauf , 2010: Linear stability analysis of Runge-Kutta based partial time-splitting schemes for the Euler equations, *MWR*

COSMO-EU (7 km): (since 29. June 2010)
Replacement of the dynamical core ('Leapfrog-scheme', *Klemp, Wilhelmson (1978) MWR*) by the 'Runge-Kutta-scheme' (*Wicker, Skamarock (2002) MWR, Baldauf (2010) MWR*)

Goal:
'convergence' of the dynamical cores of COSMO-EU and COSMO-DE

Motivation:

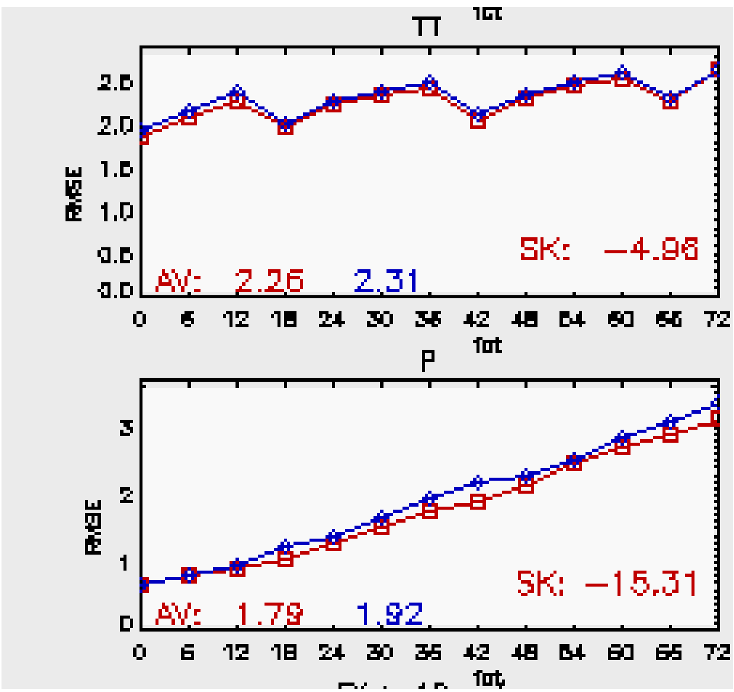
- higher accuracy of the RK-scheme towards leapfrog
  (in particular better horizontal advection for the dynamic variables);
  additionally better transport schemes for humidity variables
- maintenance: only to foster one dynamical core
- future developments are easier to do with a 2-timelevel scheme instead of a 3-timelevel scheme, e.g. physics-dynamics-coupling

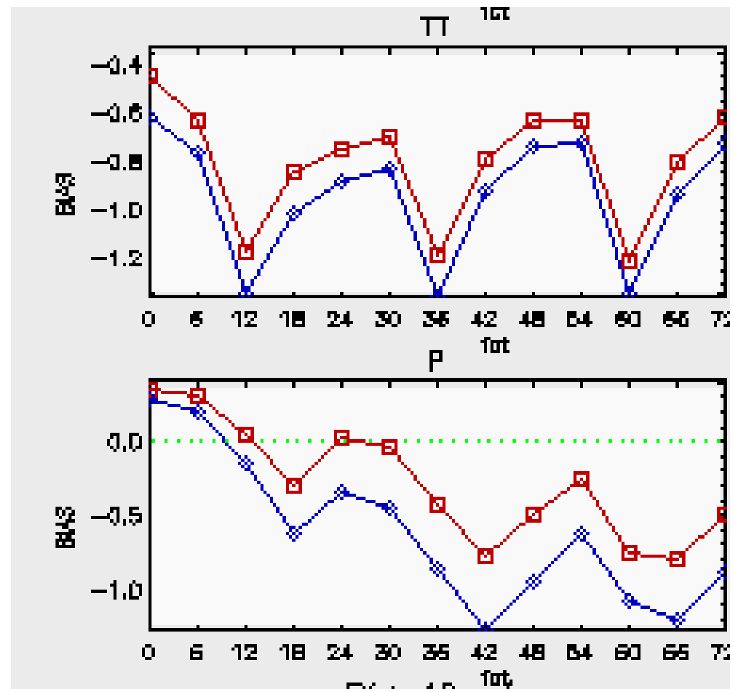
(G. Zängl, M. Baldauf, A. Seifert, J.-P. Schulz, DWD)

Measurements to reduce a pressure bias

- more accurate discretization of metrical terms (in pressure gradient) for the stretched vertical coordinate (Gal-Chen-coord.)
  (← definition: main levels geometrically are situated in the middle of the half levels)
- Introduction of a subgrid scale orography (SSO)-scheme (*Lott, Miller (1997) QJRMS*)
- use of a new reference atmosphere (allows z $\rightarrow\infty$)
- consistent calculation of base state pressure $p_0(z)$ on the main levels (i.e. not by interpolation but by analytic calculation)
- improved lower (slip-) boundary condition for w:
  upwind 3rd order + extrapolation of $\mathbf{v}_h$ to the bottom surface

SYNOP-Verification, 03.02.-06.03.2010, 0 UTC runs

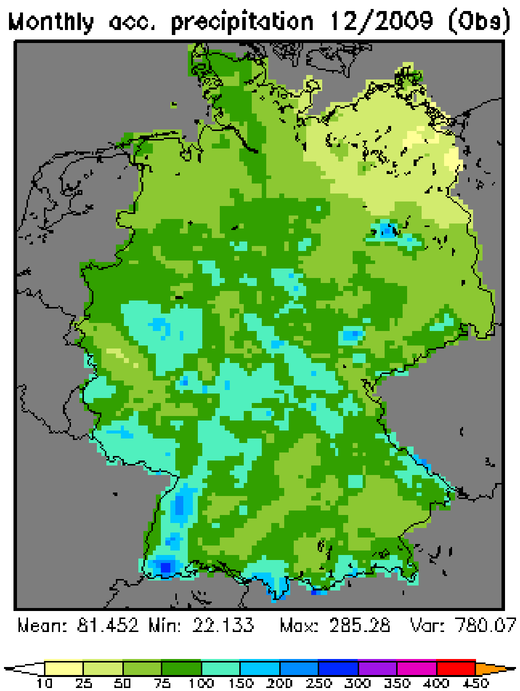




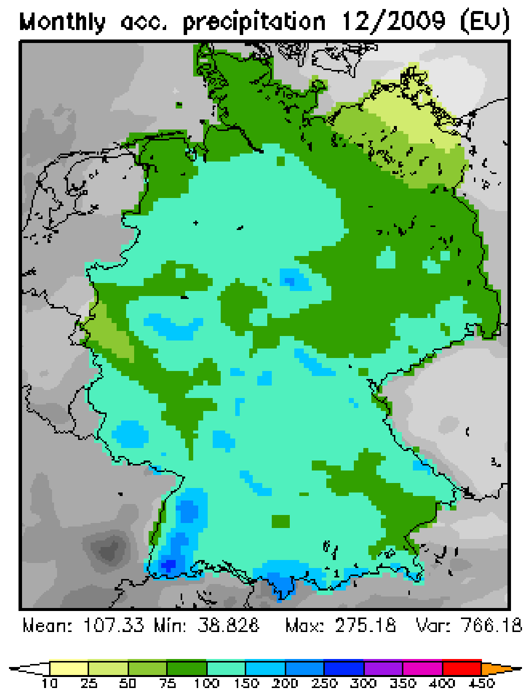
(U. Damrath) **COSMO-EU RK (new)** **COSMO-EU Leapfrog (old)**

Model climatology: monthly average of precipitation 12/2009

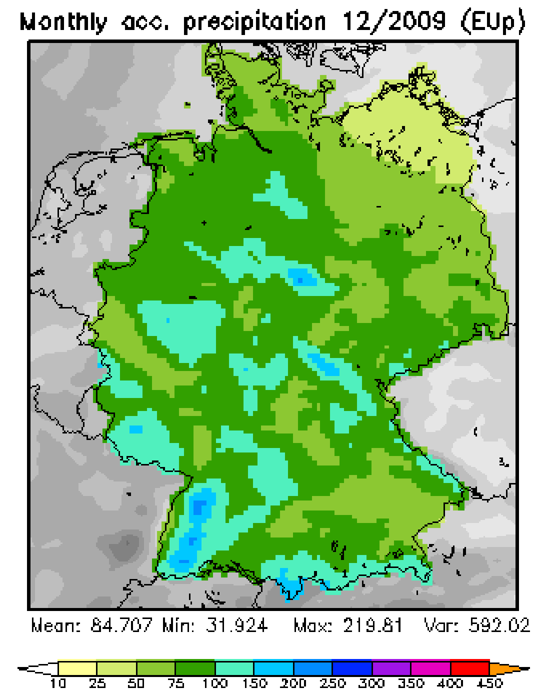
observation



COSMO-EU Leapfrog



COSMO-EU RK



(A. Seifert)

Future plans in the development of the dynamical core of COSMO

Improvements in the RK dynamical core
(motivation: 'COSMO-DE with 50 → 65 levels')

Implicit vertical advection of 3rd order

Goal: improve vertical transport of dynamical variables u, v, w, T', p' in the convection resolving COSMO-DE

- higher (3rd) order scheme
- useable for higher vertical Courant numbers
- matches better with RK philosophy (temporal order conditions)

Improvements in the fast waves solver

- Proper discretization of vertical stretching (grid staggering)
- use of divergence in conservative form

DFG-Schwerpunktprogramm 'Metström'

Projekt 'Adaptive Numerics for Multiscale Phenomena'

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Discontinuous Galerkin Method

- Seek weak solutions of a balance equation
  (correspondance to finite volume methods → conservation)
- Expand solution into a sum of base functions on each grid cell
  (correspondance to finite element methods → arbitrary high order possible)

- useable on arbitrary grids → suitable for complex geometries
- discontinuous elements → mass matrix is block-diagonal
  in combination with an explicit time integration scheme
  (e.g. Runge-Kutta → RKDG-methods) → highly parallelizable code
  but: vertically expanding sound waves need some implicitness

Cockburn, Shu (1989) Math. Comp.
Peraire, Persson (2008) SIAM J. Sci. Comp. (compact DG)

COSMO-Priority Project 'Conservative dynamical core'

Main goals:

- develop a dynamical core with at least conservation of mass, possibly also of energy and momentum
- better performance and stability in steep terrain

2 development branches:

- assess aerodynamical implicit Finite-Volume solvers (*Jameson, 1991*)
  P.L. Vitagliano (CIRA, Italy), L. Torrisi (CNMCA, Italy) M. Baldauf
- assess dynamical core of EULAG (e.g. *Grabowski, Smolarkiewicz, 2002*)
  M. Ziemianski, M. Kurowski, B. Rosa, D. Wojcik (IMGW, Poland), M. Baldauf

Atmosphere at rest (G. Zaengl (2004) MetZ)
Cold bubble (Straka et al. (1993)) (unstationary density flow)
Mountain flow tests (stationary, orographic flows)
(Schaer et al. (2002), Bonaventura (2000) , Pierrehumbert, Wyman (1985))
Linear Gravity waves (Skamarock, Klemp (1994), Giraldo (2008))
Warm bubble (Robert (1993), Giraldo (2008))
Moist, warm bubble (Weisman, Klemp (1982) MWR)

Comparison between the compressible equations and the anelastic approximation; linear analysis (normal modes)

$p = p_0 + p'$
$T = T_0 + T'$

$$
\begin{aligned}
\frac{\partial u'}{\partial t} + u_0 \frac{\partial u'}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \alpha_D \frac{\partial D'}{\partial x} \\
\delta_1 \left(\frac{\partial w'}{\partial t} + u_0 \frac{\partial w'}{\partial x} \right) &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \delta_{LH} \frac{N^2}{g}\frac{p'}{\rho_0} - g\frac{\rho'}{\rho_0} + \alpha_{D,v}\frac{\partial D'}{\partial z} \\
\delta_2 \frac{\partial \rho'}{\partial t} + \delta_3 u_0 \frac{\partial \rho'}{\partial x} + \delta_5 w' \frac{\partial \rho_0}{\partial z} &= -\rho_0 D' \\
\frac{\partial p'}{\partial t} + \delta_4 u_0 \frac{\partial p'}{\partial x} + w' \frac{\partial p_0}{\partial z} &= c_0^2 \left(\frac{\partial \rho'}{\partial t} + u_0 \frac{\partial \rho'}{\partial x} + w' \frac{\partial \rho_0}{\partial z} \right) \\
D' &:= \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}
\end{aligned}
$$

divergence damping

switches:
- compressible: $\delta_{1..5}=1$
- compr. + div. damp.: $\delta_{1..5}=1$
- anelastic: $\delta_{2,3}=0$, $\delta_{1,4,5}=1$
  OP62, WO72, compr.: $\delta_{LH}=0$
  Lipps, Hemler (1982): $\delta_{LH}=1$

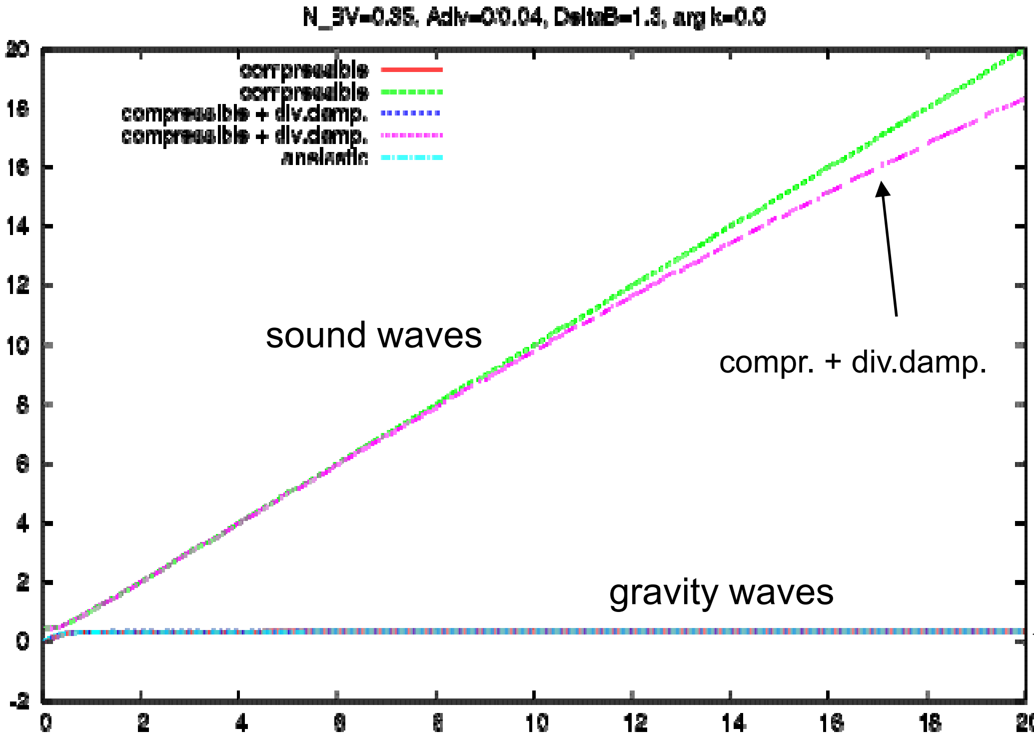
Bretherton-Transformation: $\phi' = \left(\frac{\rho_0}{\rho_s}\right)^{\alpha} \cdot \phi_b$

(inverse) scale height: $\frac{1}{H} := -\frac{\partial}{\partial z} \log \frac{\rho_0}{\rho_s}$ ~ $(10\text{ km})^{-1}$

Dispersion relation $\omega = \omega(k_x, k_z)$ of internal waves

wave ansatz: $u(x,z,t) = u(k_x, k_z, \omega) \exp(i (k_x x + k_z z - \omega t))$, $w(x,z,t) = \ldots$

Re ω/ω_a



ω_a →

← $N \cos \beta$

$k * c_s / \omega_a$

↑ λ ~ 7 km

↑ λ ~ 3.5 km

c_s *sound velocity* *(~ 330 m/s)*

$$c_s^2 = \frac{c_p}{c_v} RT$$

N Brunt-Vaisala-frequency *(~ 0.01 1/s)*

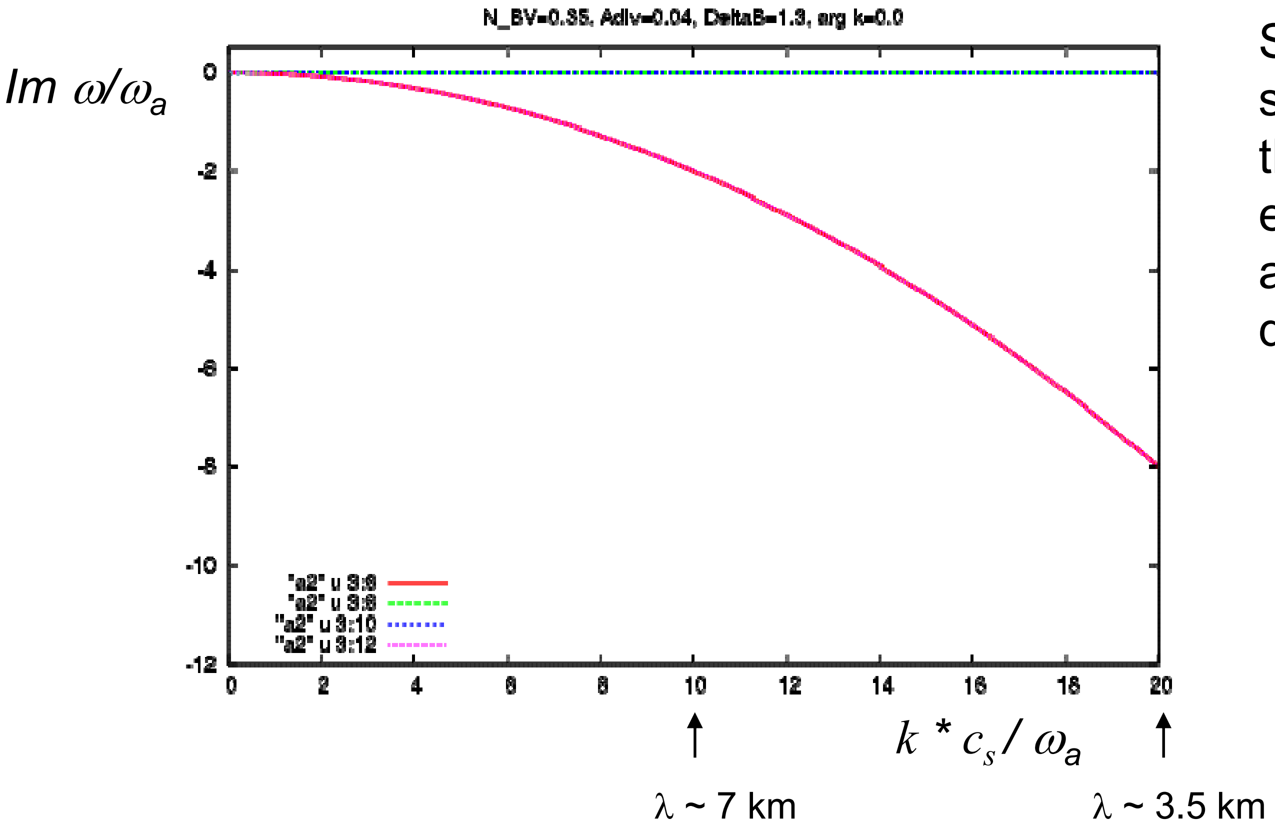
$$N^2 = \frac{g}{T} \left(\frac{\partial T}{\partial z} + \frac{g}{c_p} \right)$$

ω_a acoustic cut off frequency *(~ 0.03 1/s)*

$$\omega_a^2 = N^2 + \frac{g^2}{c_s^2}$$

$\beta = \angle (k_z, k_x) = 0°$

Dispersion relation $\omega = \omega(k_x, k_z)$ of internal waves



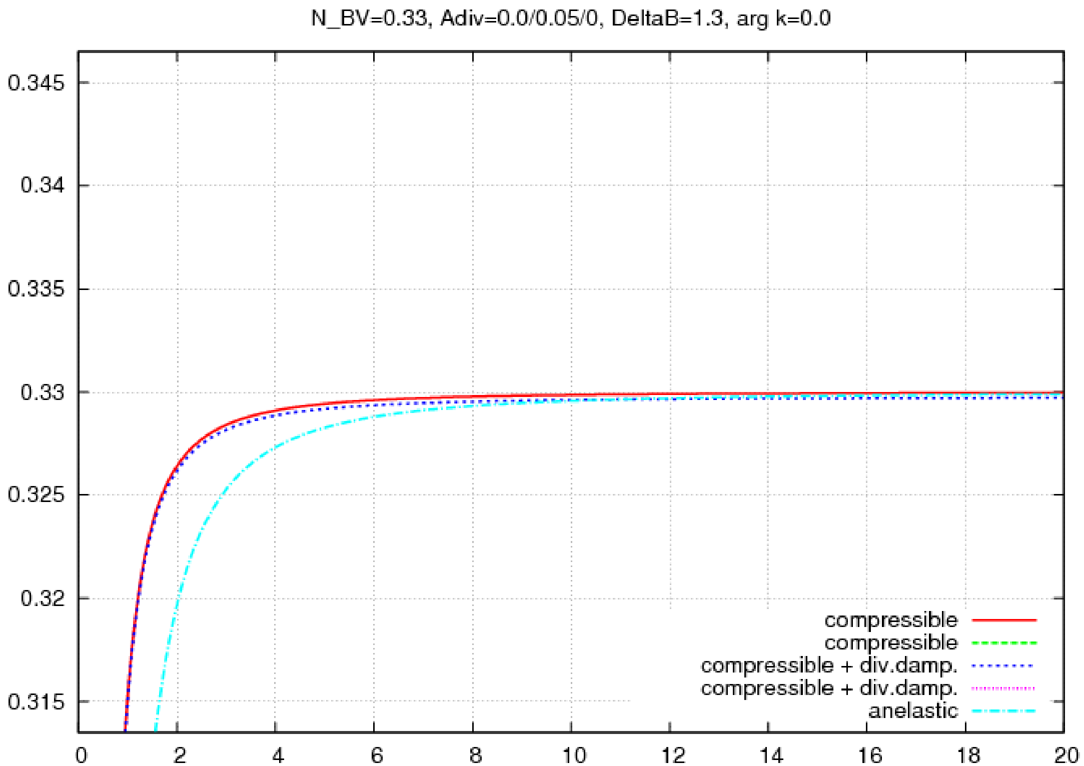
Strong damping of short sound waves in the compressible equations due to artificial divergence damping.

(timescale $1/\omega_a$ ~ 30 sec.)

N ~ 0.01 1/s
C_{div} ~0.1

Dispersion relation $\omega = \omega(k_x, k_z)$ of internal waves; only gravity waves

ω / ω_a



$k * c_s / \omega_a$

↑ λ ~ 7 km

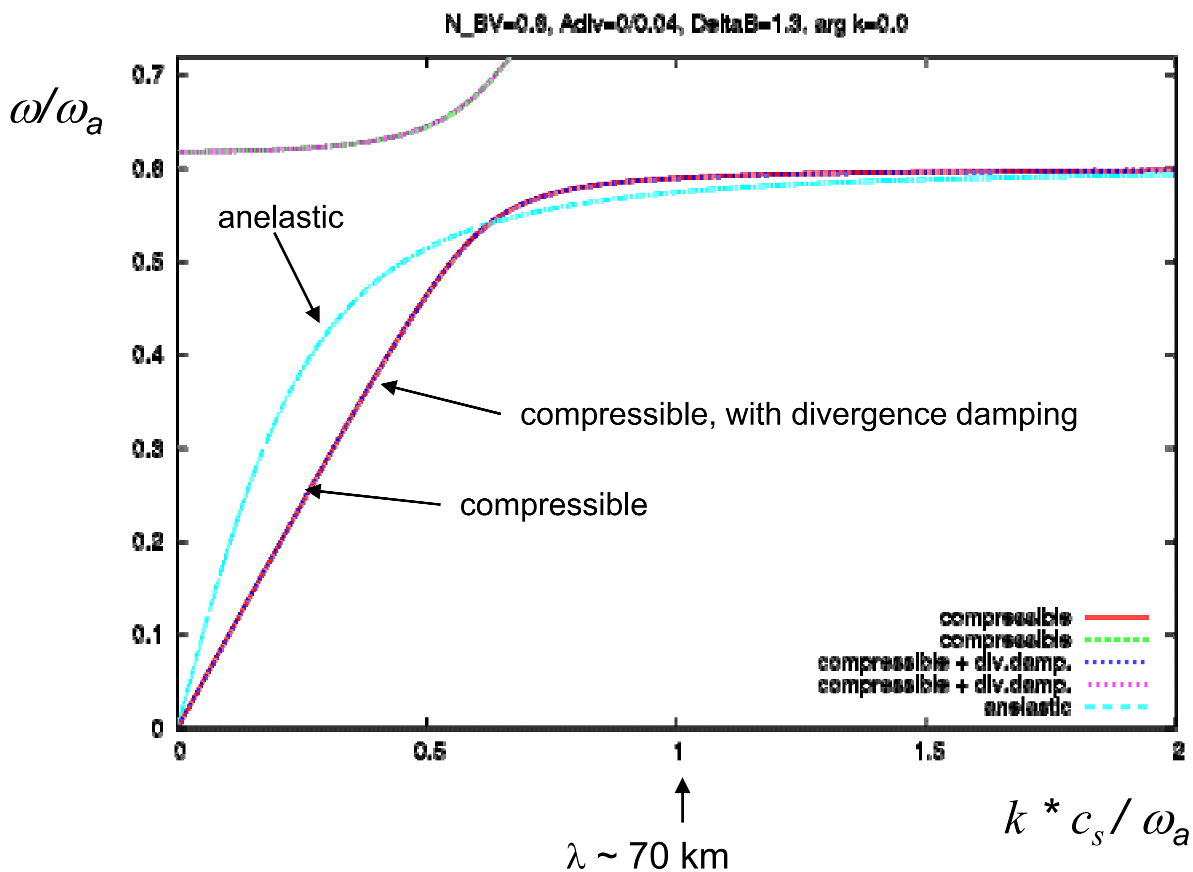
↑ λ ~ 3.5 km

$\beta = \angle (k_z, k_x) = 0°$

← $N \cos \beta$ N ~ 0.01 1/s

quite similar dispersion relation for anelastic and compressible eqns. for shorter gravity waves

Dispersion relation $\omega = \omega(k_x, k_z)$ of internal waves; focus on long gravity waves

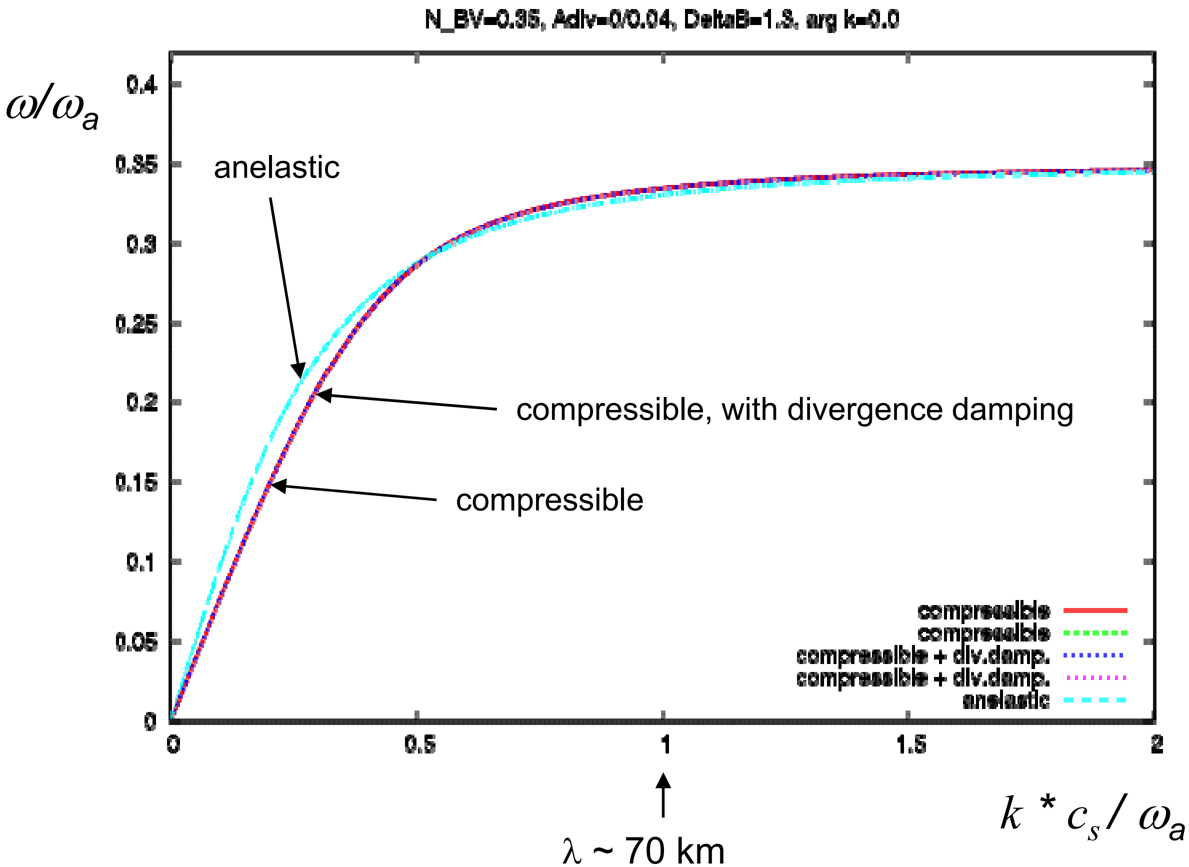


$\beta = \angle(k_z, k_x) = 0°$

$N \sim 0.018$ 1/s (isothermal)

anelastic equations of
Ogura, Phillips (1962)
Wilhelmson, Ogura (1972)

Dispersion relation $\omega = \omega(k_x, k_z)$ of internal waves
only gravity waves



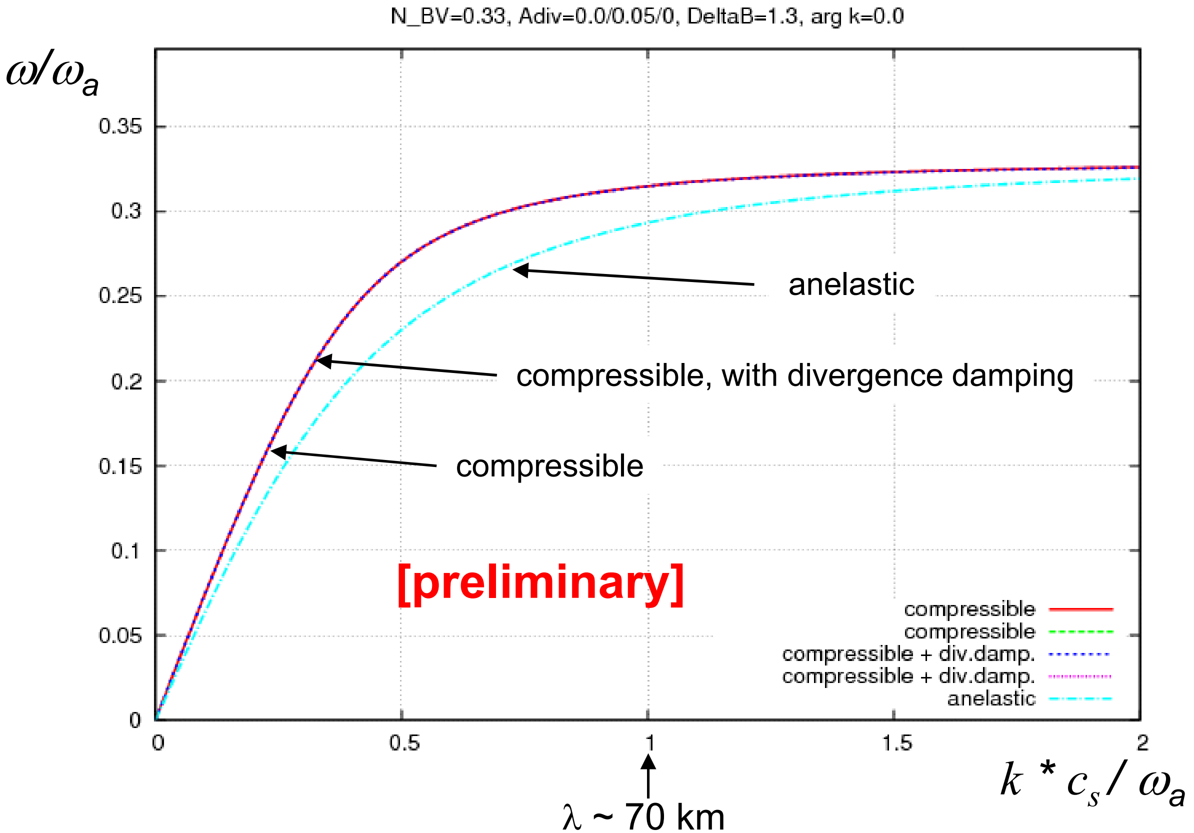
$\beta = \angle(k_z, k_x) = 0°$

$N \sim 0.01$ 1/s

anelastic equations:
Ogura, Phillips (1962)
Wilhelmson, Ogura (1972)

smaller differences for very long gravity waves

Dispersion relation $\omega = \omega(k_x, k_z)$ of internal waves; only gravity waves



$\beta = \angle(k_z, k_x) = 0°$

N ~ 0.01 1/s

anelastic equations:
Lipps, Hemler (1982)

Summary

anelastic equations seem to be well suited for small scale model applications (regional modeling, convective-permitting regime, ...,
e.g. very good performance of the EULAG model for all standard non-hydrostatic test cases)
but phase velocity of very long gravity waves is not correct.
(Error is even bigger for the Lipps, Hemler (1982) system)

Davies et al. (2003) QJRMS
'Validity of anelastic and other equation sets as inferred from normal-mode analysis'

compressible equations:
divergence damping has nearly no impact on gravity waves.
Even the real part of longer sound waves is only weakly disturbed,
but of course short sound waves are heavily damped.

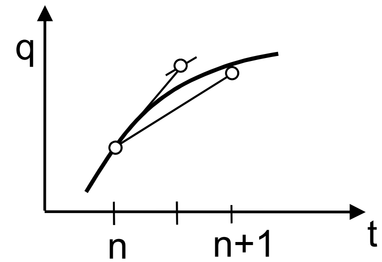
Runge-Kutta-Time-Integration

autonomous ODE-system

$$\frac{dq_l}{dt} = f_l(q_1, ..., q_M), \quad l = 1, 2, ..., M$$

explicit *N*-stage Runge-Kutta-method (to integrate from t^n to t^{n+1}):

$$\begin{aligned} q_l^{(0)} &\equiv q_l^n, \\ q_l^{(i)} &= q_l^{(0)} + \Delta t \cdot \sum_{j=1}^{i} \beta_{i+1,j} \, f_l(\mathbf{q}^{(j-1)}), \quad i = 1, 2, ..., N \\ q_l^{n+1} &\equiv q_l^{(N)} \end{aligned}$$



(other equivalent representations are possible)

Butcher-Tableau:

$$\begin{array}{c|ccccc} 0 & & & & & \\ \alpha_2 & \beta_{21} & & & & \\ \alpha_3 & \beta_{31} & \beta_{32} & & & \\ \ldots & \ldots & \ldots & & & \\ \alpha_N & \beta_{N1} & \beta_{N2} & \ldots & \beta_{N,N-1} & \\ \hline & \beta_{N+1,1} & \beta_{N+1,2} & \ldots & \beta_{N+1,N-1} & \beta_{N+1,N} \end{array}$$

$$\alpha_i := \sum_{j=1}^{i-1} \beta_{ij}$$

Consistency condition → at least 1st order accuracy:

$$\sum_{i=1}^{N} \beta_{N+1,i} = 1$$

example:
4 conditions for RK 3rd order:

$$\begin{aligned} \beta_{41} + \beta_{42} + \beta_{43} &= 1, \\ \beta_{42}\,\beta_{21} + \beta_{43}\,(\beta_{31} + \beta_{32}) &= 1/2, \\ \beta_{42}\,\beta_{21}^2 + \beta_{43}\,(\beta_{31} + \beta_{32})^2 &= 1/3, \\ \beta_{43}\,\beta_{32}\,\beta_{21} &= 1/6, \end{aligned}$$

stage N	# of β_{ij}	# of cond. for N th order RK	# of cond. for N th order LC-RK
1	1	1	1
2	3	2	2
3	6	4	3
4	10	8	4
5	15	12(?)	5
...	...	...	...

general theory:
Butcher (1964, ...),
Butcher (1987)

von Neumann-stability analysis, some helpful considerations ...

Deutscher Wetterdienst
Wetter und Klima aus einer Hand

Standard-Testproblem (Dahlquist-problem)

$$\frac{d\mathbf{q}}{dt} = \mathbf{Pq}$$

linear, homogeneous ODE-system (M equations) with time-independent M*M Matrix $\mathbf{P}$

Theorem: for the linear Standard-Testproblem, an N-stage RK-method is of **order N**, if the N conditions

$$h_{N+1}^{(l)} = \frac{1}{l!}, \qquad l = 1, 2, ..., N$$

hold. Such a scheme is called here **Linear Case-Runge-Kutta (LC-RK) of order N.**
($h_N^{(l)}$ are polynomials in the β_{ij} , they can easily be calculated by a recursion formula)

example: the ‚3rd order RK method' used for WRF *(Wicker, Skamarock, 2002)* is a 3-stage, 2nd order RK, but a 3rd order LC-RK

Lemma: all LC-RK methods of order N behave similar for the standard testproblem.

Lemma:
All RK methods with stage=order =N are a subset of the LC-RK methods of order N

→ all 'stage=order' RK methods of the same order have the same stability properties for the standard testproblem!

Baldauf (2008) JCP

Von-Neumann stability analysis

Linearized PDE-system for $u(x,z,t)$, $w(x,z,t)$, ... with constant coefficients
Discretization u^n_{jl}, w^n_{jl}, ... (grid sizes Δx, Δz)
single Fourier-Mode:

$$u^n_{jl} = u^n(k_x, k_z) \exp(i (k_x j \Delta x + k_z l \Delta z))$$

2-timelevel schemes:

$$\begin{pmatrix} u^{n+1} \\ w^{n+1} \\ p'^{n+1} \\ T'^{n+1} \end{pmatrix} = Q \begin{pmatrix} u^n \\ u^n \\ p'^n \\ T'^n \end{pmatrix}$$

Determine eigenvalues λ_i of Q
scheme is stable, if $\max_i |\lambda_i| \leq 1$

find λ_i analytically or numerically by scanning $k_x \Delta x = -\pi..+\pi$, $k_z \Delta z = -\pi..+\pi$

Sound

$$\frac{u^{n+1}-u^n}{\Delta t} = -\frac{1}{\rho_0}\left(\beta_1^s\,\delta_x p'^{n+1} + (1-\beta_1^s)\,\delta_x p'^n\right)$$
$$\frac{w^{n+1}-w^n}{\Delta t} = -\frac{1}{\rho_0}\left(\beta_2^s\,\delta_z p'^{n+1} + (1-\beta_2^s)\,\delta_z p'^n\right)$$
$$\frac{p^{n+1}-p^n}{\Delta t} = -\frac{c_p}{c_v}p_0\left(\beta_3^s\,\delta_x u^{n+1} + (1-\beta_3^s)\,\delta_x u^n + \beta_4^s\,\delta_z w^{n+1} + (1-\beta_4^s)\,\delta_z w^n\right)$$
$$\frac{T^{n+1}-T^n}{\Delta t} = -\frac{R}{c_v}T_0\left(\beta_5^s\,\delta_x u^{n+1} + (1-\beta_5^s)\,\delta_x u^n + \beta_6^s\,\delta_z w^{n+1} + (1-\beta_4^s)\,\delta_z w^n\right)$$

- temporal discret.: 'generalized' Crank-Nicholson $\beta=1$: implicit, $\beta=0$: explicit
- spatial discret. δ_x, δ_z: centered diff.

Courant-numbers: $C_{snd,x} = c_s\frac{\Delta t}{\Delta x}$, $C_{snd,z} = c_s\frac{\Delta t}{\Delta z}$ $\quad c_s^2 = \frac{c_p}{c_v}RT$

fully explicit ($\beta_1 = \ldots = \beta_6 = 0$)	uncond. unstable	-	
forward-backward (*Mesinger, 1977*), unstaggered grid	stable for $C_x^2+C_z^2<2$	neutral	4 dx, 4dz
forward-backward, staggered grid (e.g. $\beta_{3,4,5,6}=1$)	stable for $C_x^2+C_z^2<1$	neutral	2 dx, 2dz
forward-backw.+vertically Crank-Nic. ($\beta_{2,4,6}=1/2$)	stable for $C_x<1$	neutral	2 dx
forward-backw.+vertically Crank-Nic. ($\beta_{2,4,6}>1/2$)	**stable for $C_x<1$**	**damping**	**2 dx**
fully implicit ($\beta_1 = \ldots = \beta_6 = 1$)	uncond. stable	damping	

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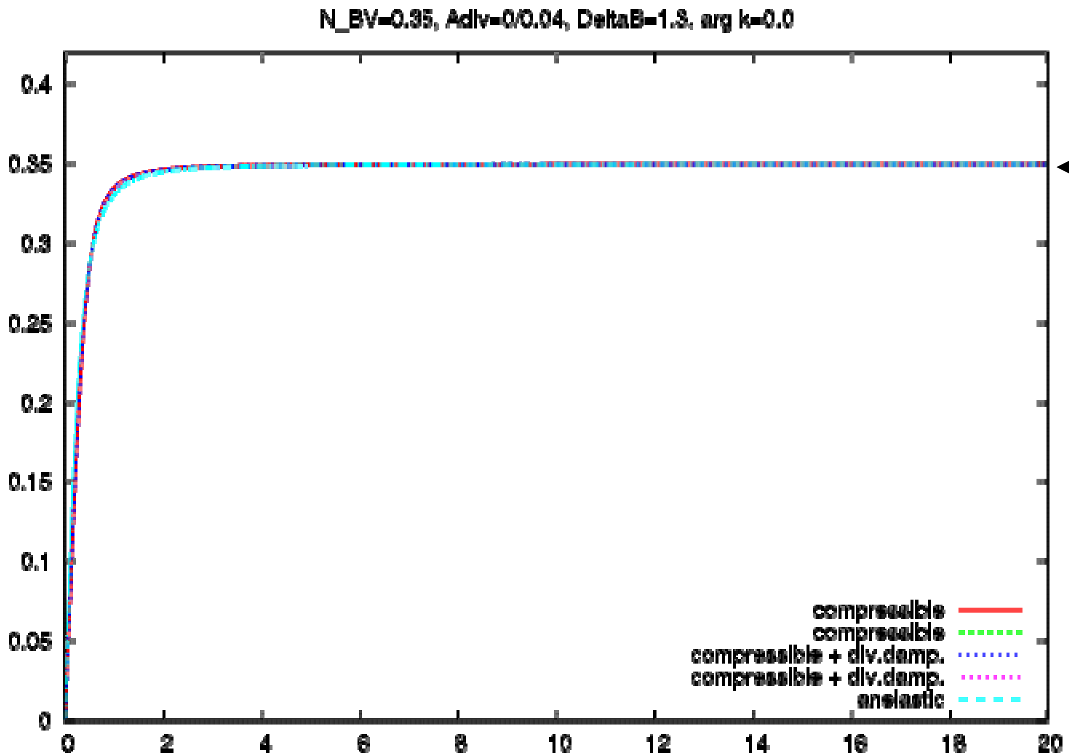
ω/ω_a



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