Efficient algorithms for non-linear four-wave interactions

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This paper addresses the on-going activities in the development of efficient methods for computing the non-linear four-wave interactions in operational discrete third-generation wind-wave models. It is generally assumed that these interactions play an important role in the evolution of wind generated surface gravity waves. Therefore, present day wave models contain parameterizations of these interactions. Despite the fact that a closed mathematical formulation is available for the description of these interactions, a full solution is not applicable in operational modelling as it requires excessive computational requirements. To overcome this limitation, various approximate methods have been developed that retain most of the basic properties of the full solution. However, the present operational parameterisations of these interactions have various weaknesses that degrade model performance and which generally lead to too broad spectra. Efficient and more accurate algorithms are therefore needed to improve model performance. As with each approximate method, a gain in computational speed goes together with a loss in accuracy in evaluating the basic integral, but this may not translate in less accurate wave model results. Therefore the definition of efficiency in the context of approximate methods for computing the four-wave interactions is addressed. Two main lines of development are identified in the development of efficient algorithms: extended Discrete Interaction Approximations, and reduced quasi-exact solution techniques.

1. Introduction

It is generally assumed that non-linear four-wave interactions play an important role in the evolution of wind generated waves. A closed formulation exists to compute in which way these interactions exchange energy within a wave spectrum. However, this method requires huge computational effort because it is written as a six-fold integral with two delta-functions ensuring conservation of wave energy, wave action and wave momentum. Because of this complicated structure, the full solution is not applicable in operational wave prediction methods. This holds especially for third-generation discrete spectral methods in which each relevant physical process is parameterised via a source term and in which no constraints are imposed on the spectrum. To achieve an operationally feasible model, many approximate methods have been developed. These approaches differ in many ways and each has its own advantages and disadvantages from a modelling point of view.

Developing a good computational method for the non-linear four-wave interactions (Snl4) requires finding a balance between computational requirements and accuracy. In this search one can distinguish two approaches: finding an efficient way to approximate the full (exact) solution of the non-linear transfer rate for a limited set of wave spectra, or develop an efficient parameterisation that, in combination with other source terms, reproduces model behaviour in comparison to a wave model including an exact method for these interactions.

The aim of this paper is twofold. Firstly, to present an overview of the present developments to efficiently compute the non-linear four-wave interactions in operational discrete spectral wave prediction models. Secondly, to discuss the concept ‘efficient’ in relation to computational requirements, accuracy, model performance and types of application.

2. Computation of non-linear four-wave interactions

2.1. Basic approach

The first description of the energy transfer between four different wave components was given by Phillips (1960). His theory of non-linear four-wave interactions was further extended to a random sea independently by Hasselmann (1962) and Zakharov (1968), and the resulting six-fold integral is known as the Boltzmann integral or kinetic equation, respectively. This integral describes the rate of change of wave action density n_i for the wave number vector $\boldsymbol{k}_1$ due to resonant interactions with three other wave number vectors $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ according to:

$$\begin{aligned}\frac{\partial n_1}{\partial t} = &\iiint G\left(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4\right)\times\delta\left(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4\right)\times\delta\left(\omega_1+\omega_2-\omega_3-\omega_4\right)\\ &\times\left[n_1n_2\left(n_3+n_4\right)-\left(n_1+n_2\right)n_3n_4\right]d\boldsymbol{k}_2d\boldsymbol{k}_3d\boldsymbol{k}_4\end{aligned} \tag{2.1}$$

in which the δ-functions reflect the resonance conditions which also ensure conservation wave energy, wave action and wave momentum. The corresponding radian frequencies ω_i are coupled with the wave number k_i via the linear dispersion relation. The term G is the coupling coefficient which is a complicated function of the four wave number vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ and depth. Expressions for this term in deep water were provided by Hasselmann (1962) and simplified by Webb (1978) and in a slightly corrected form by Dungey and Hui (1979). A shallow water version of the coupling coefficient was given by Herterich and Hasselmann (1980).

The important role of these non-linear four-wave interactions became clear in the analysis of the JONSWAP experiment (Hasselmann et al., 1973), where it was concluded that the down shifting of the peak frequency and the growth at the forward face of the spectrum during wave growth can be attributed to the nonlinear transfer to longer waves. Also, it became evident that the non-linear interactions are also responsible for shape stabilization of wind wave spectra. Non-linear four-wave interactions play a role at large and small time scales. An overview of its role in wind wave evolution can be found in e.g. Young and Van Vledder (1993).

After the JONSWAP experiment various computational methods were developed to evaluate the transfer integral and to study the properties of these interactions. These can be distinguished in analytical and numerical methods. As no analytical solution exists for the transfer rate for an arbitrary spectrum, first analyses were restricted to idealized narrow peaked spectra. Such analyses were performed by Longuet-Higgins (1976) and Fox (1976) who were able determine some basic principles of the interactions. This concept was extended by Dungey and Hui (1979) who introduced a first order approximation with respect to spectral width. These analytical expressions can be used to test numerical methods to evaluate the transfer integral, but so far this has seldom been done.

The first numerical method to evaluate the transfer integral was developed by Sell and Hasselmann (1972). Further progress was made by Hasselmann and Hasselmann (1981) who developed a symmetrical method which became the main part of the EXACT-NL model (SWAMP, 1982). This model is a one-dimensional discrete spectral wave model capable of either computing fetch- or duration limited wave growth in combination with the then state-of-the-art source terms for wind growth and white-capping. The EXACT-NL model has been used to study e.g. the existence of

equilibrium wave spectra (Komen et al., 1984), the source term balance in shallow water (Weber, 1988), and the directional response of wind waves in turning winds (Van Vledder and Holthuijsen, 1993).

2.2. Analytical methods

The mathematical structure of the Boltzmann integral (2.1) is difficult to cast in a computational method as all kinds of advanced numerical techniques need to be applied to handle the δ-functions. However, it can be rewritten in a more manageable form by analytically integrating over the δ-functions. Integrating over the δ-functions effectively reduces the six-fold integral into a three-fold integral, although the integration space is a multi-dimensional manifold in wave number space. Further, this space has to satisfy the resonance conditions or, equivalently, the conservation laws.

Analytical methods were developed by various researchers each making different choices in their analytical transformations. Three main methods exist, which mainly differ in the choice of transformation variables to eliminate the δ-functions and to handle singularities in the integration domain. The first method is due to Webb (1978), whose method describes the rate of change of action density at wave number vector $\boldsymbol{k}_1$ as a function of wave number vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$. The second method is due to Masuda (1980) whose basic transformation variables are the wave number vectors $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$. The third method is due to Lavrenov (2001) who introduced Gaussian quadrature formulas adapted to these singularities arising from the transformations.

Each of these analytical methods has been cast into a numerical solution technique. The Webb (1978) approach has been cast in the so-named WRT-method based on work of Tracy and Resio (1982). Resio and Perrie (1991) extended the WRT method to shallow water and Van Vledder (2006) made an operational version of the WRT method, which is now implemented in various third-generation wave models as an option to accurately estimate the non-linear four-wave interactions in discrete spectral wave models. The Masuda (1980) method has been cast in the RIAM (Research Institute for Applied Mathematics) method, to which Komatsu and Masuda (1986) and Hashimoto et al. (1998) have contributed. The Lavrenov (2001) method has been cast into an operational code by Gagnaire-Renou et al. (2010) and is known as the Gaussian Quadrature Method (GQM). These three quasi-exact methods are also denoted as Xnl, derived from eXact Non-Linear transfer.

As each of these methods is based on the same basic integral, one would expect that these methods yield the same non-linear transfer rate for a given spectrum. Surprisingly, an inter-comparison to check the mutual equivalence has never been done, although each developer has used computational results of Hasselmann's method to verify each approach. Therefore, a statement that these methods give the same answer cannot yet be given. Difficulties arise as each computational method contains various internal assumptions that may affect the final result. For example, the WRT contains an integration over a closed locus in wave number space and the distribution and number of grid points along the locus and choice of quadrature rule affect the final answer. The other methods also have internal methods and assumptions unknown which may affect the final outcome.

As each of these three computational methods is used as a benchmark for the development of approximation, it is all the more important that such an inter-comparison is carried out. Such an inter-comparison is now (2012) in progress by a team now consisting of G.Ph. van Vledder, M. Benoit, N. Hashimoto, H.L. Tolman and D.T. Resio. This inter-comparison follows the philosophy of the

SWAMP (1982) study and focuses firstly on computing as accurately as possible the non-linear transfer rate for a set of discrete energy density spectra with varying peakedness, directional width and spectral resolution. In this step differences in results will probably appear that might be related to the way each method is organized internally. In the next step, effects of internal switches, like handling of symmetries, treatment of the parametric tail, quadrature rules, smoothness of the wave spectrum and other assumptions will be varied to evaluate the computational requirements and accuracy of each method. Such a comparison enables making objective statements about model efficiency in relation to accuracy. In the third step, each computational method will be implemented in a simple wave prediction model, viz. supplementing it is with the same source terms for wind input and whitecapping dissipation, to study the dynamic behaviour of such a model and to assess whether the integration produces the same result and stable.

A property of Eq. (2.1) is that all interactions between resonant wave numbers in a discrete wave spectrum can be expressed as the sum of triple products of wave action densities (Snyder et al, 1993; Van Vledder, 2005). This feature makes it possible to quantify the degrees of freedom of any computational method for Snl4 by counting the number of unique triple products of energy densities at discrete frequency-direction bins (or equivalently wave number-direction bins).

3. Discrete Interaction Approximations

3.1. Initial development

The EXACT-NL model was the first third-generation wave prediction model in which no constraints were imposed on the spectral shape. Due to its computational requirements, this model, as well as the above mentioned quasi-exact methods, is not suited for operational wave predictions. This problem was more or less solved by the development of the Discrete Interaction Approximation or DIA (Hasselmann et al., 1985). This development resulted in the first operational third generation wave prediction model WAM (WAMDI, 1988). In the DIA only one set of all possible wave number configurations is used to exchange wave energy between four wave number vectors. The essential feature of the DIA is that the wave number $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are equal to each other and that the position of the wave numbers $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ are determined by a shape parameter λ. The frequencies in the DIA are related according to:

$$\begin{aligned} \sigma_1 &= \sigma_2 = \sigma \\ \sigma_3 &= (1+\lambda)\sigma \\ \sigma_4 &= (1-\lambda)\sigma \end{aligned} \tag{3.1}$$

The shape of a typical wave number configuration and its mirror image is shown in Figure 3.1 with λ=0.25. For this configuration the wave numbers vectors $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ make an angle of 11.48° and 33.56° with the wave number vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, respectively.

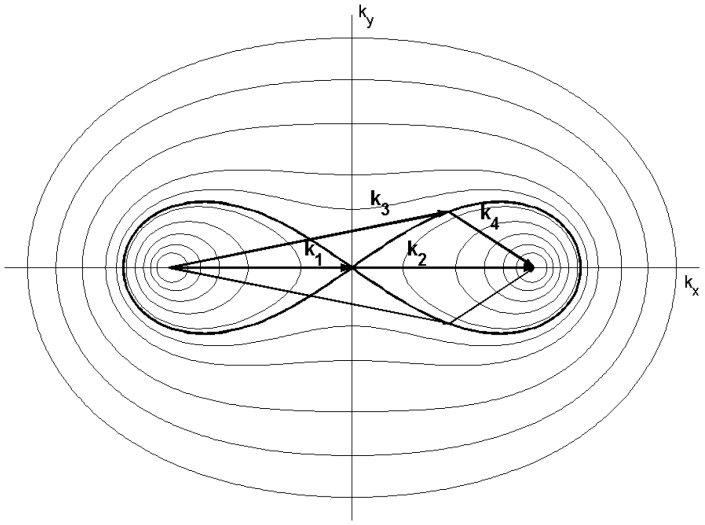

Figure 3.1: Wave number configuration of the classic Discrete Interaction Approximation with λ=0.25 and its mirror image in the interaction diagram (from Van Vledder, 2006)

The change of energy density δSnl4 at each of the four (actually three) wave number vectors is given by

$$\begin{pmatrix} \delta S_{nl,1,2} \\ \delta S_{nl,3} \\ \delta S_{nl,4} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} C_{nl4} g^{-4} f^{11} \left(E_{1,2}^2 \left(\frac{E_3}{(1+\lambda)^4} + \frac{E_4}{(1-\lambda)^4} \right) - 2E_{1,2} \frac{E_3 E_4}{(1-\lambda^2)^4} \right) \tag{3.2}$$

Here, E_i are the energy densities at the interacting wave numbers and g is the gravitational acceleration. In the DIA proposed by Hasselmann et al. (1985) the 'classic parameter values are $C_{nl4}=3\times10^6$ and $\lambda=0.25$. This choice of parameters was obtained by some testing in order to be able to reproduce as good as possible empirical growth curves of wave energy and peak period, not by attempting to reproduce the exact non-linear transfer rate for a given wave spectrum as accurately as possible. This configuration was also adopted by the WAM group (WAMDI, 1988) and used in the WaveWatch model (Tolman, 1991) and the SWAN model (Booij et al., 1999).

K. Hasselmann and S. Hasselmann (1984, personal communication) realized that a better DIA could be achieved with a multiple DIA. In fact, they originally proposed a second configuration with $\lambda=0.15$ and a weight of 3.75×10^5. This second configuration never made it to the WAM model, and subsequent third-generation models like SWAN and WaveWatch, it was realized that the added accuracy of this second configuration did not weight up against the extra computational requirements. At that time, a single DIA configuration took almost 40% of the total CPU time of a wave model run. The added accuracy of the double DIA to represent the non-linear transfer rate was partly degraded by the then state-of-the-art source terms for wind input and whitecapping dissipation. This situation forced wave modellers to tune their wave models to compensate for the errors in the parameterisations of relevant physical processes. Limitations of DIA were already know by Hasselmann et al. (1985), but its practical limitations became to become clear after years of experience (e.g. Van Vledder et al., 2000), too much transfer of wave energy towards higher frequency and too wide spectra both in frequency and in direction space. As long as their host models contain the DIA, wave model

development was hampered as errors in the DIA needed to be compensated by tuning of the other source terms. The last decades much progress has been made in the development of better source term for wave growth and decay. Therefore, better algorithms for efficiently computing the non-linear transfer rate in wind-wave spectra were needed.

Adding more λ-configurations was the next step (it is noted that the initial double DIA by Hasselmann and Hasselmann was not considered). Multiple DIA's were proposed by e.g. Van Vledder et al. (2000) and by Hashimoto and Kawaguchi (2001). The coefficients of these MDIA's were obtained by least-squares methods to minimize the error of such a DIA in comparison with the exact solution for a limited set of test spectra. At that time these MDIA's were not applied in dynamic model runs to verify whether they also provided improved model performance. This development was not successful as DIA's that are solely based on adding λ-configurations have a finite ability to approach the exact solution because they consist of only one of many possible types of wave number configurations; as can be seen in Figure 3.1, the λ-configurations all lie on one of many possible interaction curves. This exhaustion for multiple λ-based DIA's was shown by Van Vledder (2005) who showed that the number of unique triple products of energy densities reached a limit for about 6 λ-configurations while at the same time the error with respect to the exact transfer rate did not decrease anymore.

The theoretical limitations of the classic DIA were taken away by the introduction of the generalized DIA by Van Vledder (2001b) and Rasmussen (personal communication, 2001). This generalized DIA is able to represent any resonant wave number configuration by introducing two additional parameters μ and Δθ, which, together with the parameter λ, can define any wave number configuration. This extended DIA has the following relations between the frequencies of the interacting wave numbers.

$$\begin{aligned} \sigma_1 &= \sigma \\ \sigma_2 &= (1+\mu)\sigma \\ \sigma_3 &= (1+\lambda)\sigma \\ \sigma_4 &= (1-\lambda-\mu)\sigma \end{aligned} \tag{3.3}$$

The form proposed by Van Vledder (2001b) was cast into symmetric form by Tolman (2003)

$$\begin{aligned} \sigma_1 &= (1+\lambda)\sigma \\ \sigma_2 &= (1-\lambda)\sigma \\ \sigma_3 &= (1+\mu)\sigma \\ \sigma_4 &= (1-\mu)\sigma \end{aligned} \tag{3.4}$$

The angles of the interacting wave number vectors can be obtained by straightforward algebra (Van Vledder, 2006; Tolman, 2012).

3.2. Shallow water aspects

In WAMDI (1988) a simple method was proposed to include finite depth effects on the nonlinear transfer rate. Firstly, the nonlinear transfer rate is computed assuming deep water. Secondly, the resulting transfer rate is multiplied with a constant factor R.

$$S_{nl4}^{h}(f,\theta) = S_{nl4}^{\infty}(f,\theta) \times R(x) \tag{3.5}$$

This factor R is a function of the dimensionless water depth $x=k_m h$ (with h water depth and k_m a mean wave number), and constant for all spectral components of the spectrum. In this scaling the shape of the 2D non-linear transfer rate does not change, whereas in reality this shape changes. An illustration of this mismatch can be seen in Van Vledder and Bottema (2002). This mismatch in non-linear transfer rates has consequences for wave evolution in shallow water. A straightforward solution to this problem is to derive a shallow water DIA in which the interacting wave numbers satisfy the linear dispersion relationship and where the coefficient of proportionality depends on the water depth and the wave numbers in each resonant configuration as illustrated in Figure 3.2.

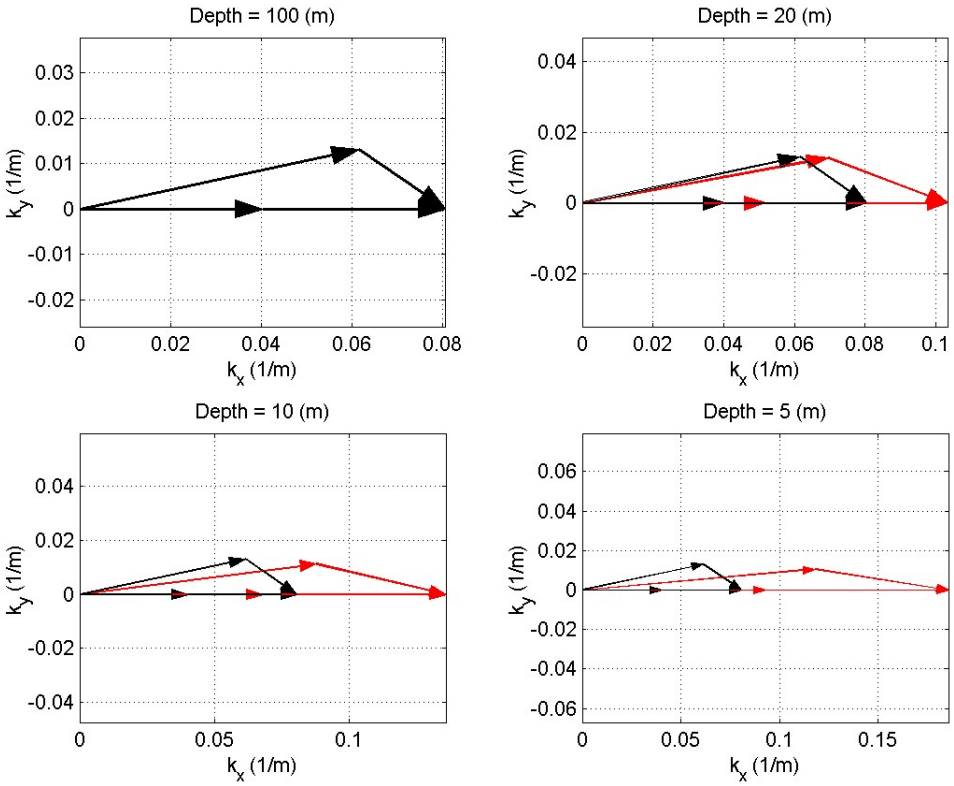

Figure 3.2: Modification of resonant wave number configuration in deep and shallow water based on a DIA wave number configuration with λ=0.25, μ=0 and $\Delta\theta$=0, for various water depths (from Van Vledder and Bottema, 2002).

Further progress was made by Tolman (2012) who derived improved scaling laws for the non-linear transfer rate in shallow water and combined it with the concept of a multiple DIA with arbitrarily shaped configuration into the Generalized Multiple DIA (GMD). The change of energy density in each of the four interaction wave numbers according to the GMD can be written as:

$$\begin{pmatrix} \delta S_{nl,1} \\ \delta S_{nl,2} \\ \delta S_{nl,3} \\ \delta S_{nl,4} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \left(\frac{1}{n_d} C_{deep} B_{deep} + \frac{1}{n_s} C_{shallow} B_{shallow} \right) \times \left(\left(\frac{c_{g1} E_1}{\sigma_1 k_1} \right) \left(\frac{c_{g2} E_2}{\sigma_2 k_2} \right) \left\{ \left(\frac{c_{g3} E_3}{\sigma_3 k_3} \right) \left(\frac{c_{g4} E_4}{\sigma_4 k_4} \right) \right\} - \left\{ \left(\frac{c_{g1} E_1}{\sigma_1 k_1} \right) + \left(\frac{c_{g2} E_2}{\sigma_2 k_2} \right) \right\} \left(\frac{c_{g3} E_3}{\sigma_3 k_3} \right) \left(\frac{c_{g4} E_4}{\sigma_4 k_4} \right) \right) \tag{3.7}$$

In which C_{deep}, C_{shallow} are coefficients of proportionality and B_{deep} and B_{shallow} are scaling functions representing weak and strong interactions, respectively, E_{i} the energy densities at the interaction wave numbers k_{i} and c_{gi} their group velocities. The numbers n_{d} and n_{s} denote the number of deep and shallow water configuration. Details of the GMD can be found in Tolman (2012).

With (3.7) a method is available to improve the DIA to any desired degree of accuracy. The problem of improving the DIA has now been reduced to finding the expansion of wave number configurations and related coefficients of proportionality. However, deriving such an extended DIA is a major challenge as no simple principle exists to make a sequence of such configurations.

3.3. Determining the coefficients of DIA's

The parameters of any DIA, including all DIA's from the classic DIA of Hasselmann et al. (1985) to the GMD of Tolman (2012), can be optimized in various ways. Initially, the shape and magnitude of DIA configurations were determined by minimizing the error between the exact non-linear transfer rate and the one from a DIA for a limited set of test spectra. This approach has various problems. The first problem is to find a set of representative test spectra, such that an optimized DIA is able to accurately represent all model spectra that might occurs in a wave model run. This goal can hardly be achieved as there are infinitely many possible spectral shapes; the number of degrees of freedom is simply too large. A second problem is to find a procedure of adding additional configurations. Such a procedure does not yet exist, also because an optimal DIA with, say, *n* configurations will probably have different configurations as a DIA with (*n*-1) configurations. A third problem is that the computational requirements of finding an optimal *n*-configuration DIA increase exponentially with the number of parameters specifying each configuration.

The above procedure may yield an optimal (multiple and/or generalized) DIA for a given set of test spectra (either academic or from measurements). This, however, is no guarantee that it will yield improved model performance in dynamic wave models runs as the performance of any wave model depends on the interplay of a DIA, as the source term for the non-linear interactions, with the other source terms for wave growth and decay. To obtain a proper wave model, also the extended DIA should be tuned, either in combination with the other source terms for a set of dynamic model runs, or only the parameters of an extended DIA. The latter approach is used in the optimization of the GMD (Tolman and Grumbine, 2012) where the optimization is carried out for a fixed set of source terms for growth and decay. The benchmark for such an optimization procedure may consist of parametric or observed growth curves or pre-computed model runs performed with an exact representation of the

non-linear four-wave interactions. Such a method is called a holistic optimization as its primary aim is to get optimal model performance. Its secondary aim is to get a proper non-linear transfer rate.

The first application of the holistic approach was by Hasselmann et al. (1985) who determined the coefficients of the single-configuration DIA such that the WAM model could more or less reproduce known growth behaviour of the total wave energy and peak frequency. Their procedure led to choosing λ=0.25 for the first configuration and λ=0.15 for the second configuration, although the latter one was never used. More detailed numerical experiments with a single configuration confirmed that the values chosen by Hasselmann et al. (1985) were close to the optimum values.

The holistic approach was applied by Tolman (2012) to determine the coefficients of various versions of the GMD, i.e. with different numbers of deep (n_d) and shallow water (n_s) configurations, for a specific set of source terms for wave growth and decay using the WAVEWATCH III™ model. First, a set of benchmark test results was created by performing dynamic model runs using the WAVEWATCH III™ model including the WRT method of Van Vledder (2006). These runs include e.g. fetch- and duration limited wave growth in deep and shallow water, the 'homogeneous' front case of Tolman (1992) and a turning wind case. The coefficients of an n_d-n_s parameter GMD were determined by a genetic algorithm (Tolman and Grumbine, 2012). Applying such a genetic algorithm is much cheaper than an explicit error mapping procedure to find an optimum configuration (Tolman and Grumbine, 2012). Another advantage of this genetic algorithm is that it may avoid getting stuck in local minima. This algorithm determines by iteration (or generation) the shape and scale parameters of a n_d-n_s GMD until a certain minimum error measure is obtained. Each generation comprises a set of dynamic wave model runs that needs to be performed.

Tolman (2012) determined various GMD's with different numbers of deep and shallow water configurations. The resulting errors in model behaviour were much smaller to the classic DIA with λ=0.25. Applying a multi-configuration GMD increases the computational requirements, but this should be considered in view of improved model performance. Tolman (2012) reports that replacing a single DIA with a GMD with three configurations, roughly requires 50% extra CPU time of the WAVEWATCH III™ model while reducing the average prediction error by 40% in comparison with a set of model runs using the WRT formulation. Adding further complexity reduces this prediction error at the expense of additional computational requirements. Note that these errors are not related to prediction errors in comparison with measurements, although applying the GMD improves the predicted spectral shapes.

The holistic optimization procedure of Tolman and Grumbine (2012) has a few drawbacks. Firstly, the results of the optimization procedure depend on the choice of accompanying source terms. There is no guarantee that it will have similar model performance in combination with other sets of source terms, but it should be noted that such tests have not been done yet. A second drawback is that the results may depend on numerical characteristic of the host model, such as spectral discretisation and the technique to solve the action balance equation. It is likely that these aspects are of minor importance and this should therefore also be tested. The third drawback may be related to the representativeness of the set of model runs, it may not yet cover a sufficient amount of conditions experienced in operational model runs. However, this limitation may also be an advantage, as the GMD may be tuned for a specific set of wave conditions, i.e. fetch-limited wave growth in deep water from a straight coast with a constant wind speed and direction. The fourth drawback is related to the

computational requirements needed for the genetic optimization. Tolman (2012) reports that about $O(10^5)$ wave model runs are required to find the parameters of a GMD with five configurations. The relevance of this drawback depends on the computer resources available at the institute developing a certain GMD using this approach.

4. Bridging the gap between Xnl and DIA's

4.1. Finding the optimum algorithm

For the computation of the non-linear four-wave interactions of a discrete wave spectrum one can choose between the accurate but time consuming quasi-exact methods like WRT, RIAM or QGM, or one may apply the fast but very inaccurate DIA. Both extremes are undesirable from a modelling point of view and many attempts have been carried out to bridge this gap by finding an optimal method for computing these interactions, i.e. finding a method that is operationally feasible while having sufficient accuracy.

In the search for an optimal algorithm two concepts play a major role. The first concept is related to the computational requirements of a certain method. This largely depends on the computer resources available. This may be less a problem for institutes like the ECMWF or NOAA/NCEP, but it may be a problem for small engineering companies who need to run a wave model for a long time or for many conditions. On the other hand, computer power is still improving in general and the limits of what's possible are still shifting. The second concept is related to accuracy. As stated before, this concept has two aspects. The first aspect is related to the ability of an approximate method to represent the exact transfer rate. The second aspect is to see an approximate method as part of a wave model in which also other source terms play a role, which together determine overall model performance. Preferably, both aspects should be of relevance at the same time. However, in the end model performance is often preferred no matter what happens inside the wave model.

The development of the GMD is an important step in bridging the gap between X_{nl} methods and the classic DIA. The results shown by Tolman (2012) are encouraging and provide the opportunity to improve model performance in general and spectral shapes in particular with a clearly defined procedure.

Comparing the WRT with a GMD with, say 4 wave number configurations, leads to an interesting observation. The number of independent wave number configurations in such a GMD is of the same order as the DIA, but still three orders of magnitude different from the WRT method applied to a typical discrete wave spectrum. In view of the overall good performance of the GMD in reproducing the non-linear transfer rate (Tolman, 2012), one has to conclude that the WRT method probably has a large amount of wave number configurations that do not contribute to the total transfer rate. This observation may hold as well for the other X_{nl} methods like GQM and RIAM. This notion suggests that in principle it should be possible to weed out unnecessary wave number configurations or combine wave number configurations in X_{nl} type of methods to improve their efficiency while retaining sufficient accuracy.

4.2. Reducing Xnl methods

As noted in the previous section, exact methods take a large number of wave number configurations into account in evaluating the non-linear transfer rate for a given discrete wave spectrum. Based on the performance of the GMD (Tolman, 2012), it is likely that many of those configurations are not necessary. Reducing the computational requirements of exact methods can be achieved in various ways. Van Vledder (2006) suggests a number of approaches to reduce the workload of the WRT method by applying mathematical or numerical methods. Similar approaches may hold as well for the RIAM and GQM. Taking advantage of the specific inner workings of the WRT method, savings are possible by reducing the number of points in the evaluation of the locus integral or applying higher order quadrature rules for the locus integration; by replacing bi-linear interpolation by a nearest bin approach to find the energy density (or action density) at the resonant wave number vectors; or by applying filtering assuming that with increasing distance in wave number space between the basic wave number vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$ the contribution to the total transfer rate decreases sufficiently to ignore these parts of the integration space.

The advantage of such methods is that the reduction of the workload is done in a mathematically consistent way independent of the characteristics of the wave spectra on which it is applied. Because of this property, competitive reduced versions of Xnl-type methods may be developed that provide a generic solution to improve model performance. Such an approach is generic in the sense that such a method does not depend on the other source terms in the wave model, or on the set of wave model runs needed to derive a GMD. A possible approach to achieve this goal is presented in the next section based on the WRT method.

4.3. Equivalence of WRT and DIA

The WRT method is able to compute the non-linear transfer rate to any desired degree of accuracy. In the WRT method the rate of change of action density n_i at wave number $\boldsymbol{k}_1$ can be expressed as:

$$\frac{\partial n_1}{\partial t} = \iint k_3 dk_3 d\theta_3 T\left(\boldsymbol{k}_1, \boldsymbol{k}_3\right) \tag{4.1}$$

In which T is a one-dimensional integral along a closed loop in wave number space

$$T\left(\boldsymbol{k}_1, \boldsymbol{k}_3\right) = \oint_s ds \times G\left(s\right) \times J\left(s\right) \times N_{1,2,3,4}\left(s\right) \tag{4.2}$$

Where s is the local variable along the locus, G(s) is the coupling coefficient, J(s) the Jacobean term and $N_{1,2,3,4}$(s) is the wave number product. Details can be found in Van Vledder (2006). Closer inspection of the integration along the locus reveals that for a given combination of the wave number vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$, discrete points of the loci for $\boldsymbol{k}_2$ and $\boldsymbol{k}_4$ form individual resonant wave number configurations. This is illustrated in Figure 4.1 for one set of points on these loci. The left panel shows the position of the arbitrarily chosen wave number vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$, together with the locus for $\boldsymbol{k}_2$ (black curve) and the locus of $\boldsymbol{k}_4$ (red curve). The right panel shows one of the possible wave number configurations. This particular wave number configuration can be considered as an arbitrarily shaped DIA-type configuration for which the shape parameters λ, μ and $\Delta\theta$ can be determined. Next the strength of this interaction can be deduced from piece wise integration along the locus.

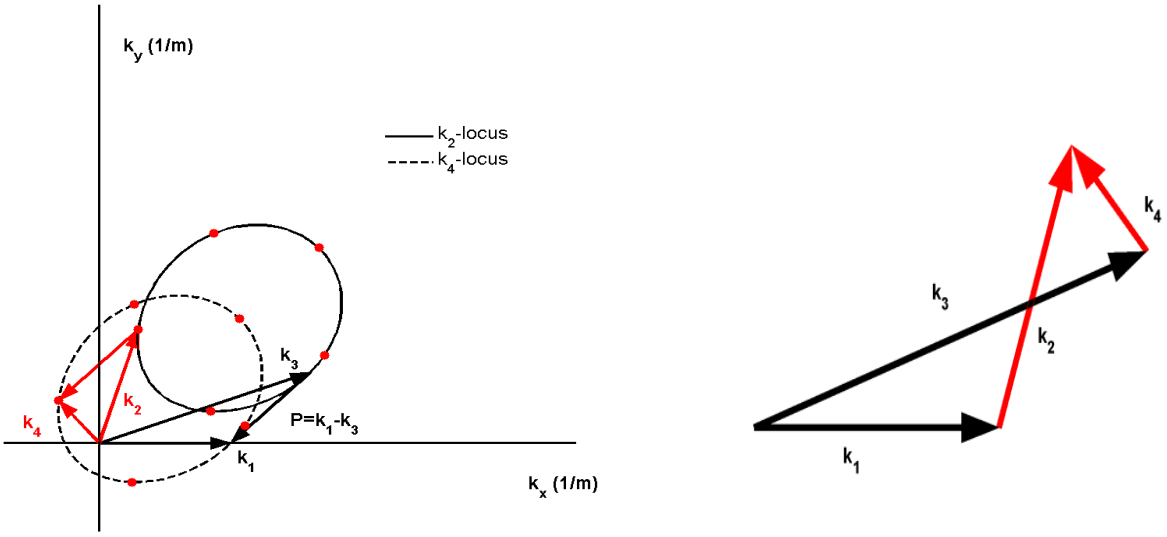

Figure 4.1: Wave number configuration based on discretisation of the integration along a locus in wave number space

In the WRT method changes are only made to each of discrete wave number vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$, based on the results of the integration along the locus. In the DIA changes are made simultaneously to all four wave numbers involved in a wave number configuration. However, due to various symmetries in the transfer (Boltzmann) integral they produce basically the same results. The principle of detailed balance forms the connection between both methods.

4.4. Other approaches

Apart from Xnl methods and extended DIA's many other types of approximate methods have been developed to compute the non-linear transfer rate. Full descriptions of these methods are not given here, and the reader is referred to the associated references. The overview is probably incomplete and no conclusions can be drawn about the order in which these methods are mentioned.

- Neural networks have been explored by e.g. Krasnopolsky et al., 2002, 2003; Tolman and Krasnopolsky (2004), Tolman et al, (2005) and Wahle et al. (2009).
- Resio and Perrie (2009) and Perrie and Resio (2010) developed the Two-Scale Approximation (TSA) in which an arbitrary spectrum is divided into a broad scale spectrum for which the exact non-linear transfer rate can be pre-computed with the WRT method, and into a residual spectrum whose contribution to the total transfer rate is approximated.
- Diffusion approaches were explored by Zakharov and Pushkarev (1999), Pushkarev et al. (2004), and Jenkins and Phillips (2001).
- Komatsu (1996) made a reduced version of the RIAM method in which 20 basic wave number configurations were taken into account. This SRIAM method is similar to a multiple DIA.
- Perrie et al. (2010) develop the Advance Dominant Interaction (AvDI) by selecting part of the phase space in WRT type methods. The selection applied results from a neural network to determine which part of the basic locus integrals should be taken into account.

All of these methods have specific advantages to estimate the non-linear transfer rate to a certain degree. However, in the author's opinion they are not flexible enough to be able to compute this transfer rate to any degree of accuracy for an arbitrary spectrum.

5. Discussion

In the last decade much progress has been made in developing operational methods for computing the non-linear transfer rate in discrete spectral wave prediction models (cf. Van Vledder, 2001a). Starting from the classical DIA with only one wave number configuration, new developments have been made in developing multiple generally shaped wave number configurations and incorporating shallow water effects. In this line of development, the GMD is presently the most advanced method available, although its applicability may be limited in some ways. Starting from exact methods operational versions have been developed like the WRT, GQM and RIAM method for academic studies (e.g. Gagnaire-Renou et al., 2010). These exact methods are also used as benchmarks for developing approximate methods. These three exact methods are all based on the same basic integral but their practical equivalence has not yet been tested. However, an inter-comparison study is now under way to objectively compare their performance in relation to computational requirements and resulting accuracy.

Shallow water effects are now included in the computational methods on the basis of the work of Herterich and Hasselmann (1980), either in the coupling coefficient in the exact methods or in parameterised form in the DIA methods. However, these parameterisations may change in view of recent insights into the transfer rate in intermediate and shallow water by e.g. Janssen and Onorato (2007) who show that the transfer rate reduces to zero for kh=1.363. The implications of these insights on wave evolution in inter-mediate depth are now being explored. The present basic equations for computing the non-linear four-wave interactions (Eq. 2.1) are based on the assumption of a homogeneous and stationary wave field. In practical applications these assumptions are often violated. Gramstad and Stiassnie (2012) investigated the effect of near-resonant conditions in non-stationary conditions as this may lead to faster evolution than on the 'Hasselmann time scale'.

6. Conclusions

The following conclusions can be formulated regarding the ongoing development of efficient algorithms for the computation of non-linear four-wave interactions in operational discrete spectral wave prediction models:

- The concept "efficient algorithm" must primarily be considered in relation to model performance and computational requirements, not only in relation to its ability to approximate the exact non-linear transfer rate to any desired degree of accuracy;
- Model efficiency should also be considered in view of the envisaged type of model application, as one may develop algorithms dedicated to a specific application;
- The GMD is a good bottom-up candidate for inclusion in an operational wave model, but it may still have limited ability due to its dependence on the set of other source terms and set of model runs for which it has been developed. These possible limitations should be determined;
- Reduced Xnl based top-down approaches may provide a generic solution to efficiently compute the non-linear interactions for discrete wave spectra;

- The differences between the quasi-exact methods (WRT, RIAM and GQM) must be objectively charted to be able to draw conclusions about their performance, both in terms of computational requirements as well as in terms of their accuracy;
- The consequences of modulational instabilities in intermediate water depth and the effects of near-resonant conditions on wave evolution need further attention.

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