Scale-dependent time integration and thermodynamic consistency for weakly compressible flows
... or ...

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin

Towards a "very balanced" compressible flow solver

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin

Thanks to ...

Ulrich Achatz	(Goethe-Universität, Frankfurt)
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Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

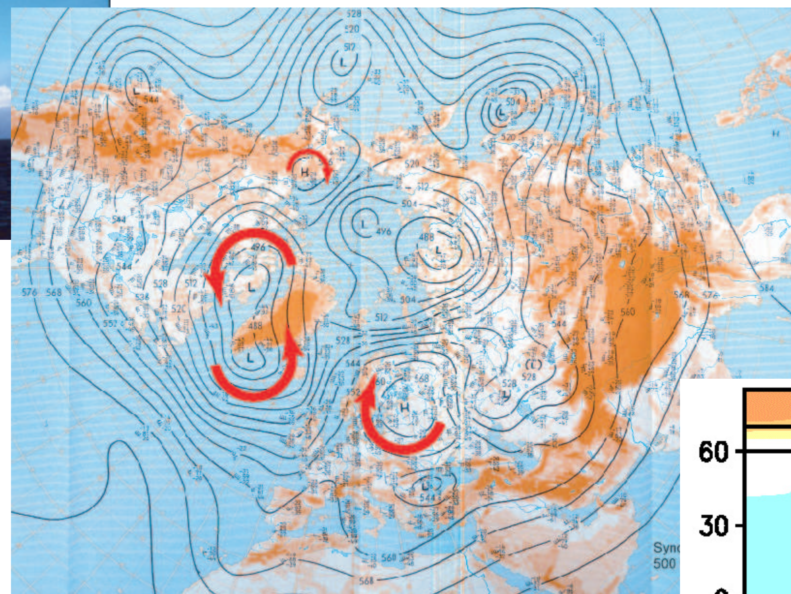
Scale-dependent time integration

Extensions: Moisture & general Eqs. of State

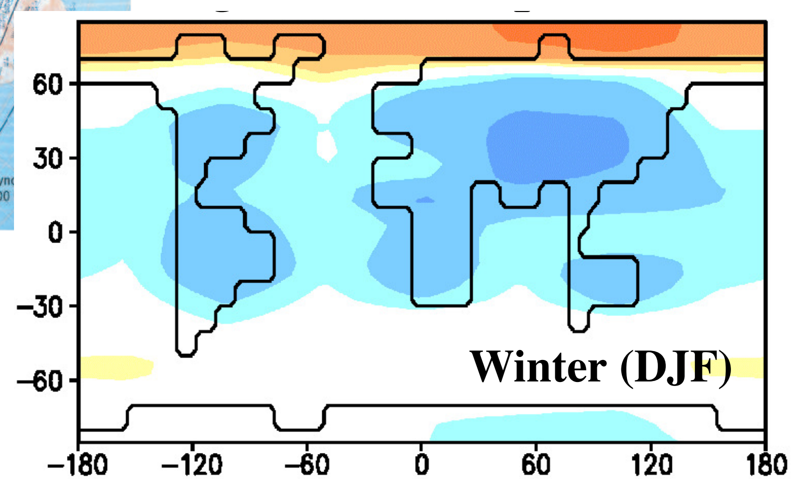
R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

Asymptotic Modelling Framework

10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Thanks to:
P.K. Taylor, Southampton Oceanogr. Inst.; P. Névir, Freie Universität Berlin; S. Rahmstorf, PIK, Potsdam

Asymptotic Modelling Framework

$$
\begin{aligned}
\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi &= \boldsymbol{S}_{\boldsymbol{u}} \\
w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z &= -\theta' + S_w \\
\theta'_t + \boldsymbol{u}\cdot\nabla\theta' + w\theta'_z &= S'_\theta \\
\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z &= 0
\end{aligned}
$$

$$
\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)
$$

Anelastic Boussinesque Model

10 km / 20 min

$$
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla\right) q = 0
$$

$$
q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)
$$

$$
\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega_0}\boldsymbol{k}\times\nabla\pi^{(3)}
$$

Quasi-geostrophic theory

1000 km / 2 days

$$
\begin{aligned}
\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T &= S_T \\
\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q &= S_q
\end{aligned}
$$

$$
Q_\varphi = \int_{z_s}^{H_a} \rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a} \rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right)dz\,, \qquad \left(\varphi\in\{T,q\}\right)
$$

$$
T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\left(\min(z,H_T) - z_s\right), \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z-z_s}{H_q}\right)
$$

$$
\rho = \rho_*\exp\left(-\frac{z}{h_{\mathrm{sc}}}\right), \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{\mathrm{sc}}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z \frac{T}{T_*}\,dz'
$$

$$
\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0
$$

V. Petoukhov et al., *CLIMBER-2 ...*, Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Asymptotic Modelling Framework

Earth's radius	a	$\sim$	$6 \cdot 10^6$	m
Earth's rotation rate	Ω	$\sim$	10^{-4}	s^{-1}
Acceleration of gravity	g	$\sim$	9.81	ms^{-2}
Sea level pressure	p_{ref}	$\sim$	10^5	$kgm^{-1}s^{-2}$
H_2O freezing temperature	T_{ref}	$\sim$	273	K
Tropospheric potential temperature variation	$\Delta\Theta$	$\sim$	40	K
Dry gas constant	R	$\sim$	287	$m^2s^{-2}K^{-1}$
Dry isentropic exponent	γ	$\sim$	1.4	

Distinguished limit:

$$\Pi_1 = \frac{h_{\text{sc}}}{a} \sim 1.6 \cdot 10^{-3} \sim \varepsilon^3$$

$$\Pi_2 = \frac{\Delta\Theta}{T_{\text{ref}}} \sim 1.5 \cdot 10^{-1} \sim \varepsilon$$

$$\Pi_3 = \frac{c_{\text{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1} \sim \sqrt{\varepsilon}$$

where

$$h_{\text{sc}} = \frac{RT_{\text{ref}}}{g} = \frac{p_{\text{ref}}}{\rho_{\text{ref}} g} \sim 8.5\,\text{km}$$

$$c_{\text{ref}} = \sqrt{RT_{\text{ref}}} = \sqrt{g h_{\text{sc}}} \sim 300\ \text{m/s}$$

distinguished limit continued

$$
\begin{aligned}
\mathrm{Fr}_{\mathrm{int}} &\sim \varepsilon \\
\mathrm{Ro}_{h_{\mathrm{sc}}} &\sim \varepsilon^{-1} \\
\mathrm{Ro}_{L_{\mathrm{Ro}}} &\sim \varepsilon \\
\mathrm{Ma} &\sim \varepsilon^{3/2}
\end{aligned}
$$

R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

Asymptotic Modelling Framework

Compressible flow equations with general source terms

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \boldsymbol{v}_{\shortparallel} + \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\shortparallel} + \frac{1}{\varepsilon^3 \rho} \nabla_{\|} p = \boldsymbol{S}_{\boldsymbol{v}_{\shortparallel}} ,$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) w + \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\perp} + \frac{1}{\varepsilon^3 \rho} \frac{\partial p}{\partial z} = S_w - \frac{1}{\varepsilon^3} ,$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \rho + \rho \nabla \cdot \boldsymbol{v} = 0 ,$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \Theta = S_\Theta .$$

Expansions

$$\begin{pmatrix} \rho \\ \boldsymbol{v}_{\shortparallel} \\ \rho \\ \Theta \end{pmatrix} =: \mathbf{U} = \sum_{i=0}^{m} \left(\varepsilon^{\alpha}\right)^{i} \mathbf{U}^{(i)} + o\left(\left(\varepsilon^{\alpha}\right)^{m}\right)$$

R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

Asymptotic Modelling Framework

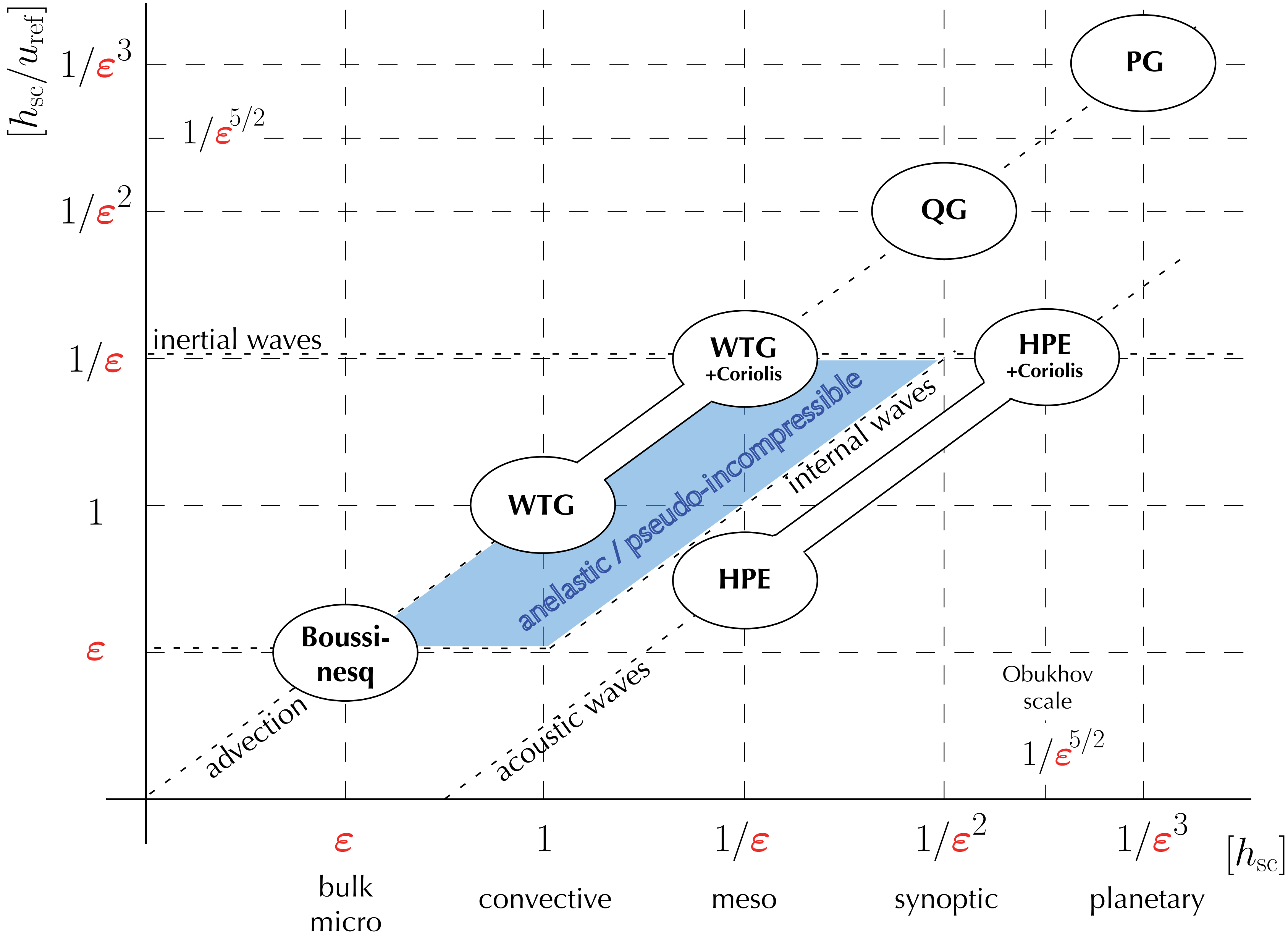
Recovered classical single-scale models:

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$	Linear small scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$	Anelastic & pseudo-incompressible models
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$	Linear large scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Mid-latitude Quasi-Geostrophic Flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Equatorial Weak Temperature Gradients
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1} \xi(\varepsilon^2 \boldsymbol{x}), z)$	Semi-geostrophic flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\underline{\varepsilon^{3/2}} t, \underline{\varepsilon^{5/2}} x, \underline{\varepsilon^{5/2}} y, z)$	Kelvin, Yanai, Rossby, and gravity Waves

... and many more

R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

Asymptotic Modelling Framework



R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State

 Klein, TCFD, **23**, 161–195, (2009); Klein, Achatz, Bresch, Knio, Smolarkiewicz, JAS, **67**, 3226–3237 (2010)

Key question:

What is the slow flow limiting dynamics like?

i.e.

What should a compressible solver do in the limit?

Sound-Proof Models

Compressible & sound-proof flow equations

$$\boldsymbol{\rho_t} + \nabla\cdot(\rho\boldsymbol{v}) \;=\; 0$$

$$(\rho\boldsymbol{u})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{u}) + P\nabla_{\|}\pi \;=\; 0$$

$$(\rho w)_t \;+\; \nabla\cdot(\rho\boldsymbol{v}w) + P\pi_z \;=\; -\rho g$$

$$\boldsymbol{P_t} + \nabla\cdot(P\boldsymbol{v}) \;=\; 0$$

drop term for:

anelastic† (approx.)

pseudo-incompressible*

(hydrostatic-primitive)

$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u}\cdot\boldsymbol{k} \equiv 0)$$

 † e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982) * Durran, JAS, **46**, 1453–1461 (1989)

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Parameter range & length and time scales of asymptotic validity ?

 † e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)   * Durran, JAS, **46**, 1453–1461 (1989)

From here on ε is the (isothermal) Mach number

$$\varepsilon = \frac{u_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}} = \frac{u_{\mathrm{ref}}}{\sqrt{g h_{\mathrm{sc}}}}$$

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

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sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips, JAS, **19**, 173–179 (1962)

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

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Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \qquad \Rightarrow \qquad \Delta\overline{\theta}\,\Big|_{z=0}^{h_{\text{sc}}} < 1\ \text{K}$$

* Ogura & Phillips, JAS, **19**, 173–179 (1962)

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{gh_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Design Regime (10 km / 20 min)

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_{\vartheta} + \tilde{w}\,\frac{d\overline{\theta}}{dz} = 0$$

$$\tilde{\boldsymbol{v}}_{\vartheta} + \frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \overline{\theta}\nabla\pi^{*} = 0$$

$$\varepsilon^{\mu}\,\pi^{*}_{\vartheta} + \left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^{*} \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^{*} \\ \boldsymbol{U}^{*} \\ W^{*} \\ \Pi^{*} \end{pmatrix}(z)\ \exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

K., Achatz, Bresch, Knio, Smolarkiewicz, JAS, **67**, 3226–3237 (2010)

Design Regime (10 km / 20 min)

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\bar{\theta}\,\bar{P}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\bar{\theta}\,\bar{P}}W^* = \frac{1}{\omega^2}\frac{\lambda^2N^2}{\bar{\theta}\,\bar{P}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^2}=O(1)\right)$

- pseudo-incompressible modes/EVals = compressible modes/EVals + $O(\varepsilon^{\mu})$ †
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

The anelastic and pseudo-incompressible models remain relevant for stratifications

$$\frac{1}{\bar{\theta}}\frac{d\bar{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{\rm sc}} \lesssim 40\ \mathrm{K}$$

not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

 † rigorous proof with D. Bresch

K., Achatz, Bresch, Knio, Smolarkiewicz, JAS, **67**, 3226–3237 (2010)

Key question:

What is the slow flow limiting dynamics like?

i.e.

What should a compressible solver do in the limit?

Answer:

Behave pseudo-incompressibly !*

* Anelastic "looses" only for breaking of internal wave packets in the stratosphere

* Achatz, Senf, K., JFM, **663**, 120–147, (2010); K., Proc. ECMWF Nonhydrostatic Modelling, (2011)

Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State

pseudo-incompressible $\Leftrightarrow$ **compressible**

Compressible

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \; = \; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \pi \; = \; -\rho g \boldsymbol{k}$$

$$\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) \; = \; 0$$

$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \quad \Gamma = c_p/R\,, \quad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)$$

Pseudo-incompressible

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \;=\; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi \;=\; -\rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) \;=\; 0$$

$\rho\theta = \overline{P}\,,$ $\qquad \pi$: “elliptic pressure”

Predictor-corrector scheme*

for

pseudo-incompressible flow

* effectively a projection method: Almgren et al. , Astrophys. J., **684**, 449–470, (2008); K., TCFD, **23**, (2009)

Predictor

Solve auxiliary hyperbolic system over $t^{\boldsymbol{n}} \longrightarrow t^{\boldsymbol{n}+1}$
*(by your favorit 2nd order scheme)**

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \; = \; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) \; = \; -\rho g \boldsymbol{k} - P \nabla \pi^{\boldsymbol{n}}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) \; = \; 0$$

Predicted values satisfy

$$\begin{pmatrix} \rho \\ P \\ \theta \end{pmatrix}^{\boldsymbol{n}+1,*} = \begin{pmatrix} \rho \\ P \\ \theta \end{pmatrix}^{\boldsymbol{n}+1} + O\left((\Delta t)^{\mathbf{3}}\right)$$

* K., TCFD, **23**, 161–195, (2009); any Runge-Kutta / Finite Volume; EULAG's-predictor …

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$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

But

$$\boldsymbol{v}^{\boldsymbol{n}+1,*} = \boldsymbol{v}^{\boldsymbol{n}+1} + O\left((\Delta t)^{\mathbf{2}}\right)$$

$$P^{\boldsymbol{n}+1,*} \neq \overline{P}$$

Corrector for advective fluxes

$$\pi^{\boldsymbol{n}+1} = \pi^{\boldsymbol{n}} + \boldsymbol{\delta\pi}$$

$$\underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}} = (P\boldsymbol{v})^{\boldsymbol{n}+1/2,*} - \frac{\Delta t}{2} P\theta \, \nabla \boldsymbol{\delta\pi}$$

$$P^{\boldsymbol{n}+1} = P^{\boldsymbol{n}} - \Delta t \, \nabla \cdot \underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}} \overset{!}{=} \overline{P}$$

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$$P^{\boldsymbol{n}+1} = P^{\boldsymbol{n}+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (P\theta \, \nabla \boldsymbol{\delta\pi}) \stackrel{!}{=} \overline{P}$$

Solve elliptic pressure equation

$$\nabla \cdot (P\theta \, \nabla \boldsymbol{\delta\pi}) = \frac{2}{(\Delta t)^2} \left(\overline{P} - P^{\boldsymbol{n}+1,*} \right)$$

Corrector for advective fluxes

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Flux correction for advected scalars $\boldsymbol{X} \in \{1, 1/\theta, \boldsymbol{v}/\theta\}$

$$(P\boldsymbol{X})^{\boldsymbol{n}+1} = (P\boldsymbol{X})^{\boldsymbol{n}+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (\boldsymbol{X} \, P\theta \, \nabla \boldsymbol{\delta\pi})$$

That's it up to ...

divergence control for $\boldsymbol{v}^{\boldsymbol{n}+1}$

some "bells & whistles"

Predictor-corrector scheme

for

compressible flow

Predictor*

Solve auxiliary hyperbolic system over $t^{\boldsymbol{n}} \longrightarrow t^{\boldsymbol{n}+1}$
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*Identical to psinc-predictor !

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$$P^{\boldsymbol{n}+1} = P^{\boldsymbol{n}} - \Delta t \nabla \cdot \underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}}$$

But now

$$P^{\boldsymbol{n}+1} - P^{\boldsymbol{n}} = \left(\frac{\partial P}{\partial \pi}\right)^{\boldsymbol{n}+1/2} \boldsymbol{\delta\pi} + O\left((\boldsymbol{\delta\pi})^3\right)$$

Corrector for advective fluxes

$$\pi^{\boldsymbol{n}+1} = \pi^{\boldsymbol{n}} + \boldsymbol{\delta\pi}$$

$$\underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}} = (P\boldsymbol{v})^{\boldsymbol{n}+1/2,*} - \frac{\Delta t}{2} P\theta \, \nabla \boldsymbol{\delta\pi}$$

$$P^{\boldsymbol{n}+1} = P^{\boldsymbol{n}} - \Delta t \, \nabla \cdot \underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}}$$

Solve Helmholtz equation

$$\frac{2}{(\Delta t)^2} \left(\frac{\partial P}{\partial \pi} \right)^{\boldsymbol{n}+1/2} \boldsymbol{\delta\pi} - \nabla \cdot \left(P\theta \, \nabla \boldsymbol{\delta\pi} \right) = \frac{2}{(\Delta t)^2} \left(P^{\boldsymbol{n}+1,*} - P^{\boldsymbol{n}} \right)$$

Corrector for advective fluxes

$$\pi^{\boldsymbol{n}+1} = \pi^{\boldsymbol{n}} + \boldsymbol{\delta\pi}$$

$$\underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}} = (P\boldsymbol{v})^{\boldsymbol{n}+1/2,*} - \frac{\Delta t}{2} P\theta \, \nabla \boldsymbol{\delta\pi}$$

$$P^{\boldsymbol{n}+1} = P^{\boldsymbol{n}} - \Delta t \, \nabla \cdot \underline{(P\boldsymbol{v})^{\boldsymbol{n}+1/2}}$$

Exner pressure post-correction

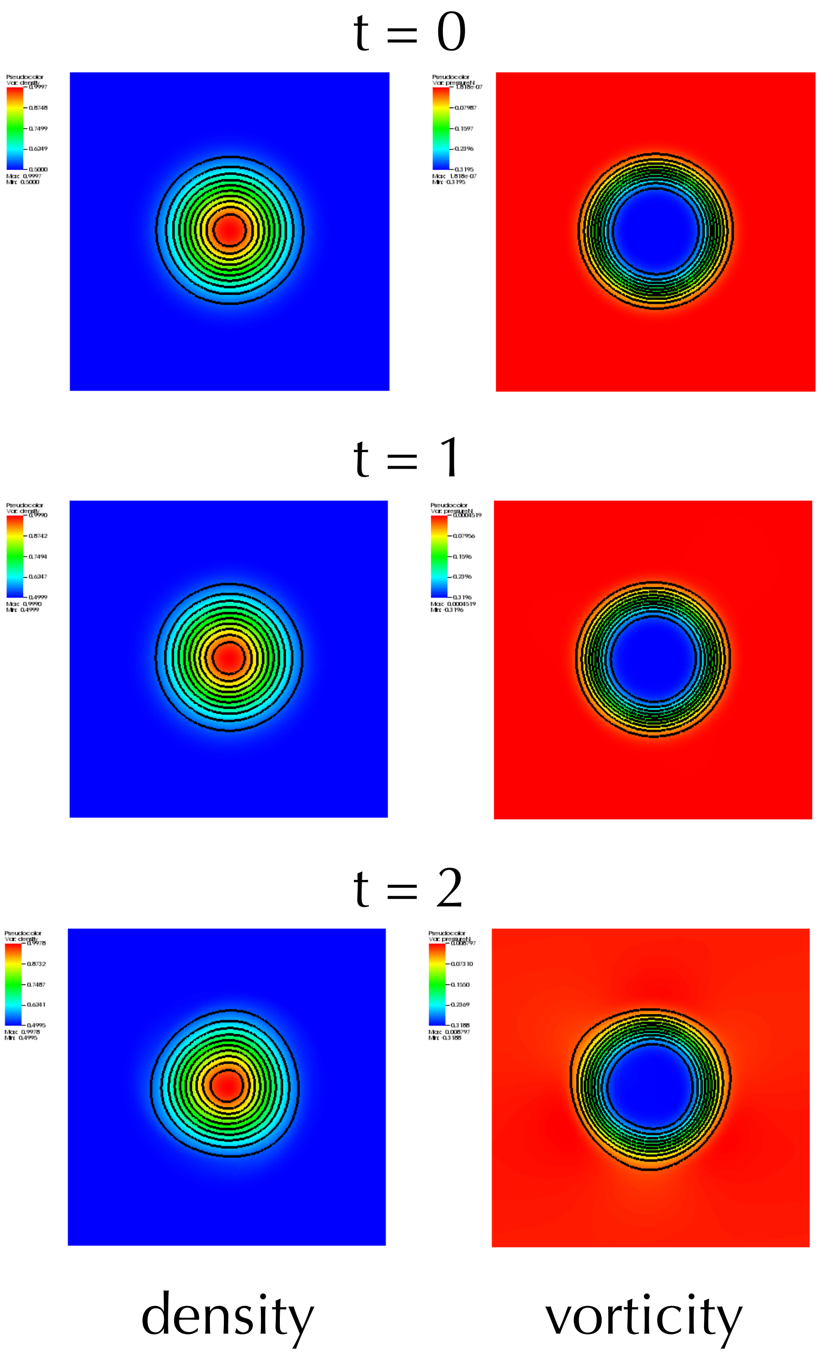
$$\pi^{\boldsymbol{n}+1} = \frac{1}{\Gamma} \left(P^{\boldsymbol{n}+1} \right)^{\gamma-1}$$

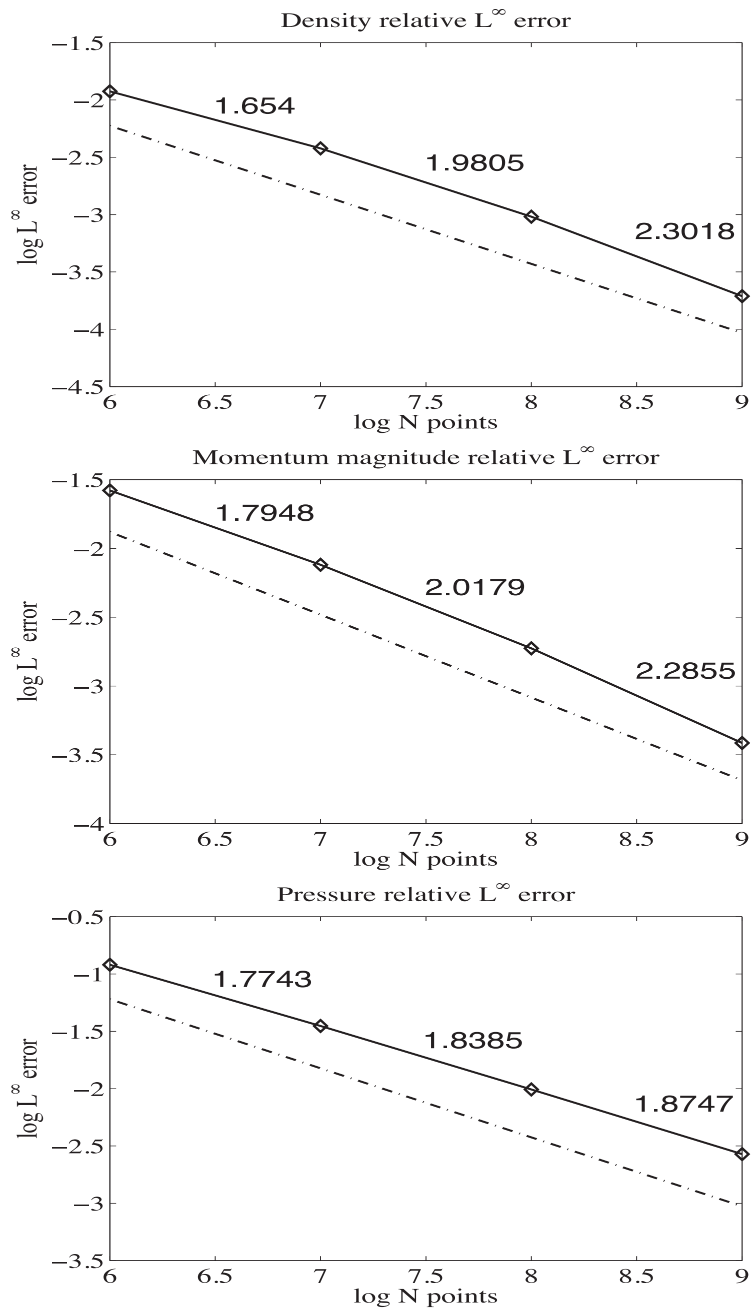
Bells & Whistles

- Well-balanced discretization of gravity term / no background state
  ([1] Botta et al., JCP, **196***, 539-565, (2004))*
- Positivity of advection in spatial op-split mode
  ([2] K., TCFD, **23***, 161–195, (2009))*
- Runge-Kutta, MUSCL-type, BDF2 predictor time integrators available
  ([2], [3] O'Neill, K., Atmos. Res., accepted, (2013), [4] Benacchio, K., t.b.p., (2013))
- Inf-Sup-stable version of projection step
  ([5] Vater, K., Num. Math., ***113****, 123-161, (2009)*

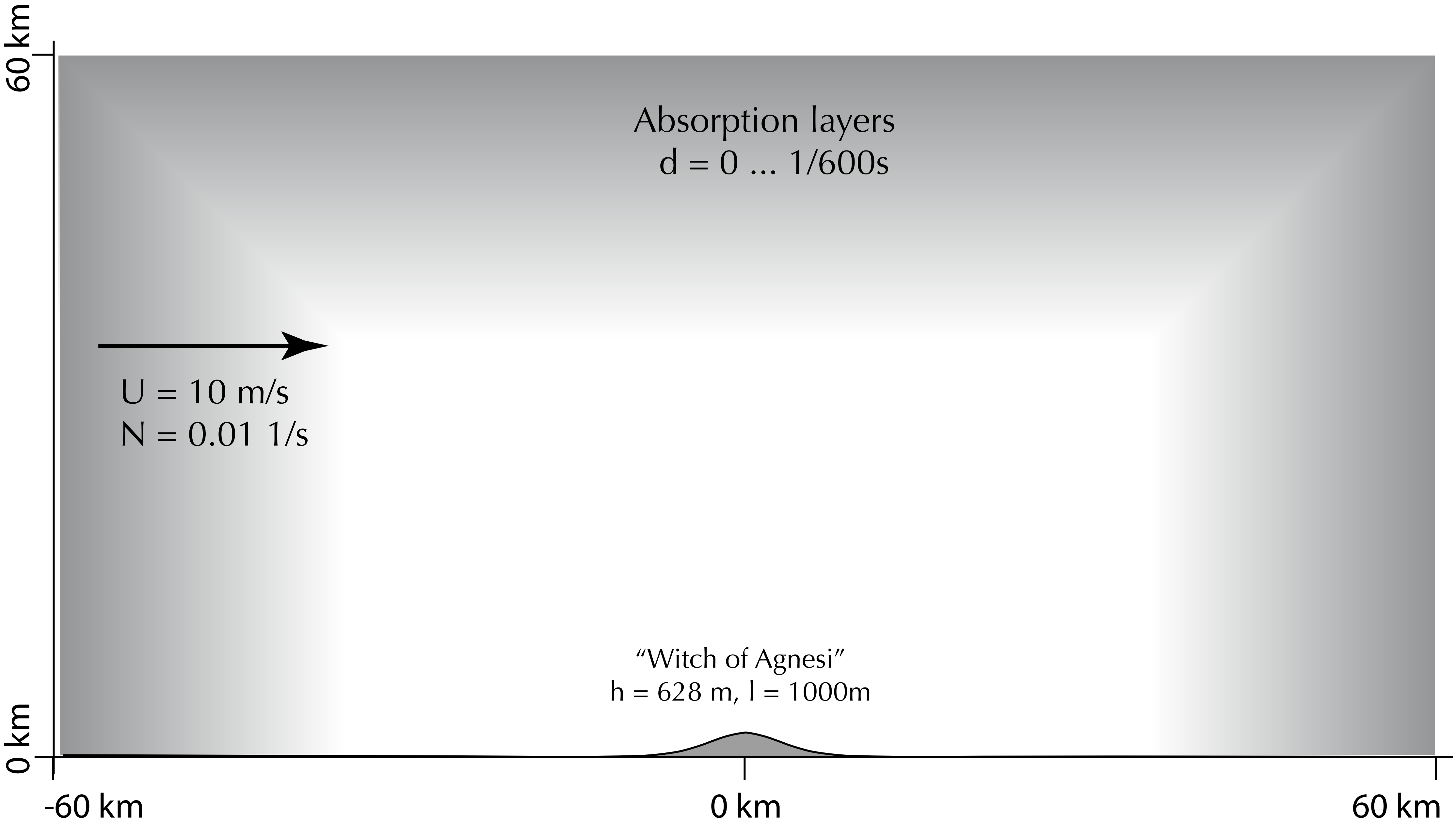
Some results

Diagonally advected vortex



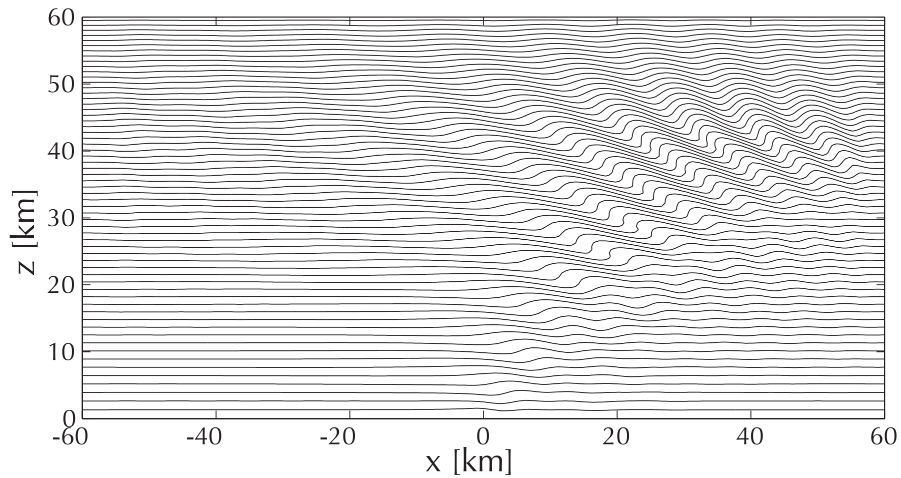


Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997)) ⇒ **Joana's talk!**

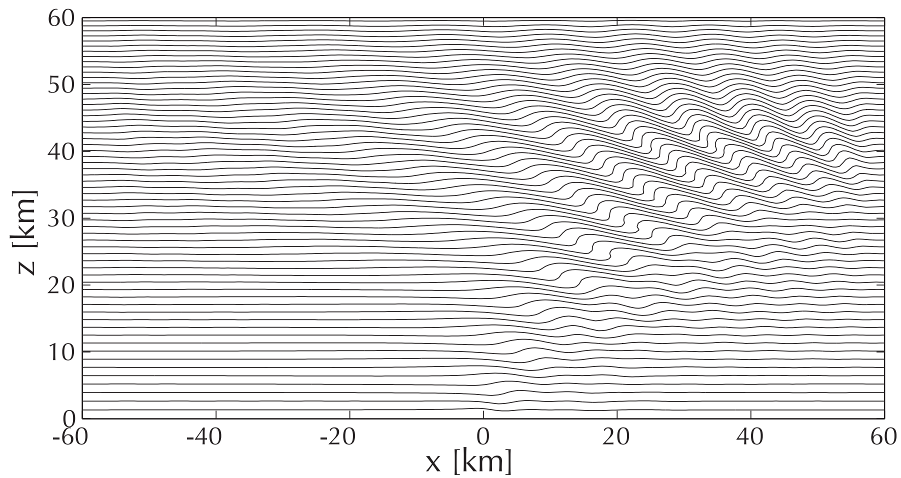


Results at time $t = 2\,h$

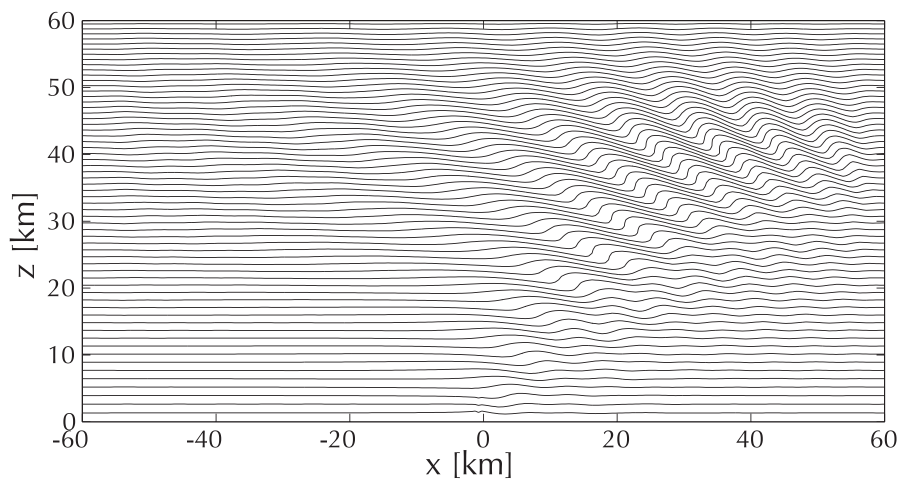
pseudo-incompressible



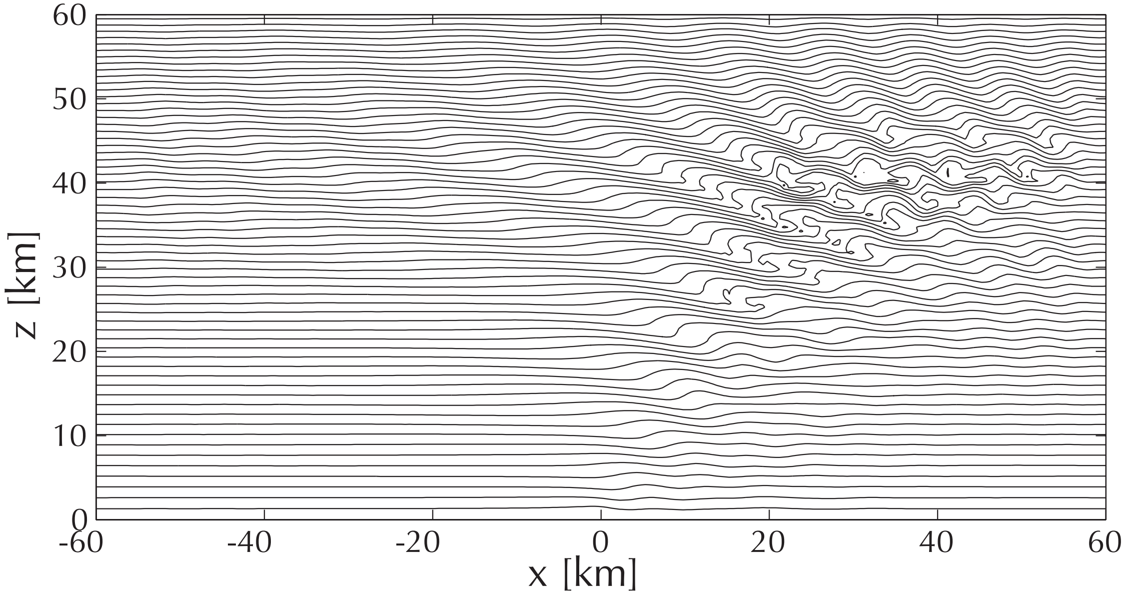
compressible, $\mathrm{CFL_{adv}} = 1$



compressible, $\mathrm{CFL_{ac}} = 2$



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



pseudo-incompressible

3 hours

sharpened van Leer's limiter

$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-4}$

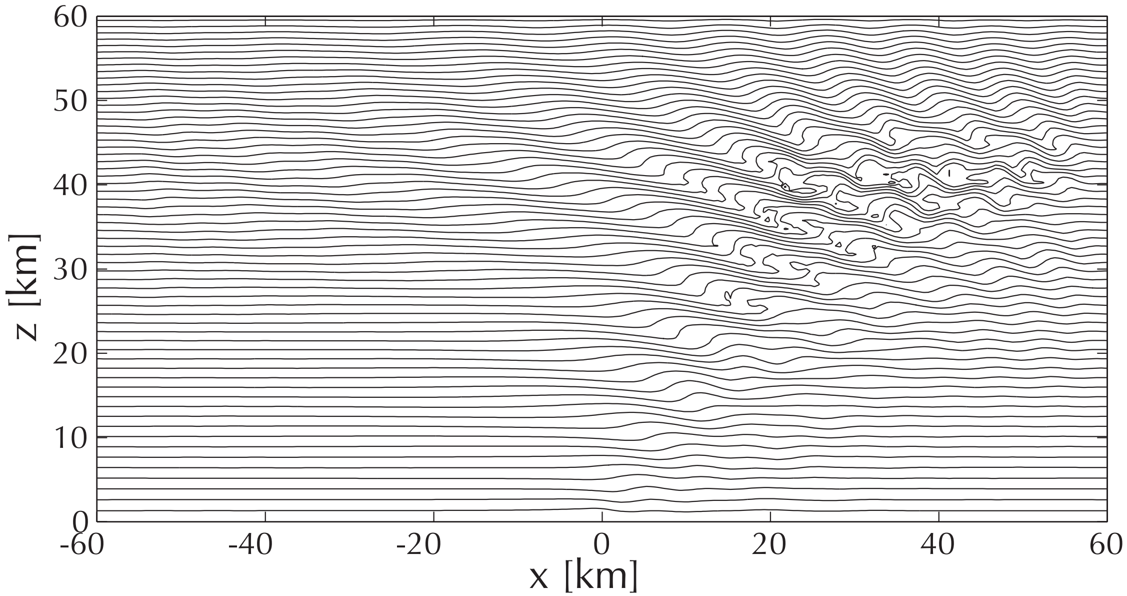
Compressible Euler eqs.

3 hours

sharpened van Leer's limiter

$\Delta t \cdot \mathrm{residual} < 10^{-4}$

$\mathrm{CFL}_{\mathrm{adv}} = 1.0$



Thermodynamically consistent "psinc"

Standard model in conservative form*

$$\rho^*_t + \nabla \cdot (\rho^* \boldsymbol{v}) \; = \; 0$$

$$(\rho^* \boldsymbol{u})_t + \nabla \cdot (\rho^* \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi \; = \; 0$$

$\Rightarrow$ advective form

$$(\rho^* w)_t \; + \nabla \cdot (\rho^* \boldsymbol{v} w) + \rho^* \theta \pi_z \; = \; -\rho^* g$$

$$\nabla \cdot (\overline{P} \boldsymbol{v}) \; = \; 0$$

$$\rho^* \theta = \overline{P} \, , \qquad \pi = \overline{\pi}(z) + \pi' \, , \qquad \boldsymbol{v} = \boldsymbol{u} + w \boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)$$

ρ^* is Durran's "pseudo-density"

* K., TCFD, **23**, 161–195, (2009)

Standard model in momentum-advective form*

$$\rho_t^* + \nabla \cdot (\rho^* \boldsymbol{v}) \; = \; 0$$

$$\boldsymbol{u}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \theta \nabla_{\|} \pi' \; = \; 0$$

$$w_t \; + \boldsymbol{v} \cdot \nabla w + \theta \pi'_z \; = \; g \frac{\theta - \overline{\theta}}{\overline{\theta}} = g \frac{\theta'}{\overline{\theta}}$$

$$\nabla \cdot (\overline{P} \boldsymbol{v}) \; = \; 0$$

ρ^* is the density effective in the momentum equation!

* Durran, JAS, **46**, 1453–1461, (1989)

Thermodynamically consistent* model in conservative form

$$\rho^*_t + \nabla \cdot (\rho^* \boldsymbol{v}) = 0$$

$$(\rho^* \boldsymbol{u})_t + \nabla \cdot (\rho^* \boldsymbol{v} \circ \boldsymbol{u}) + \nabla_{\|} p = 0$$

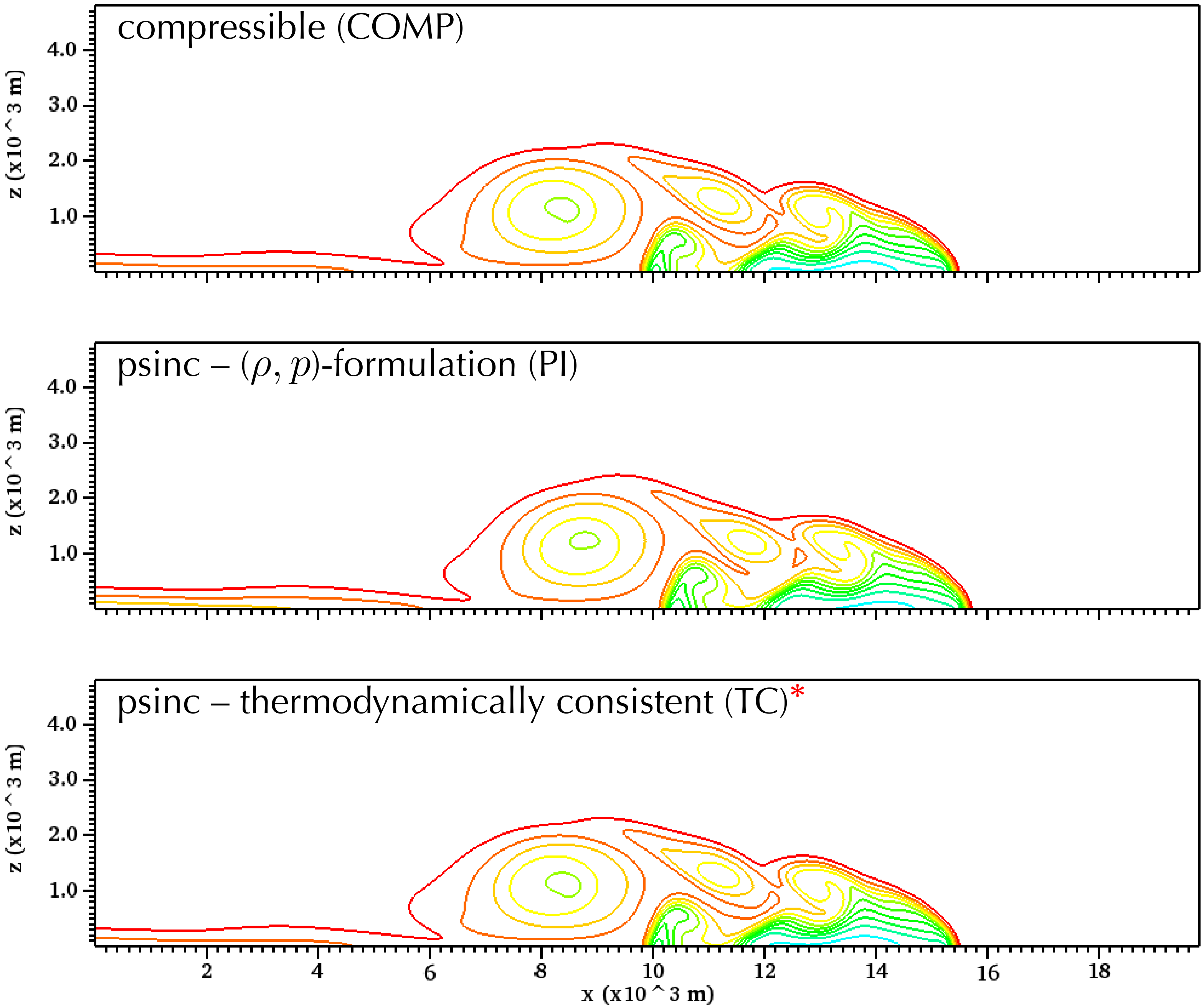
$$(\rho^* w)_t + \nabla \cdot (\rho^* \boldsymbol{v} w) + p_z = -\left(\rho^* + \frac{\partial \rho}{\partial p} p'\right) g$$

$$\nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

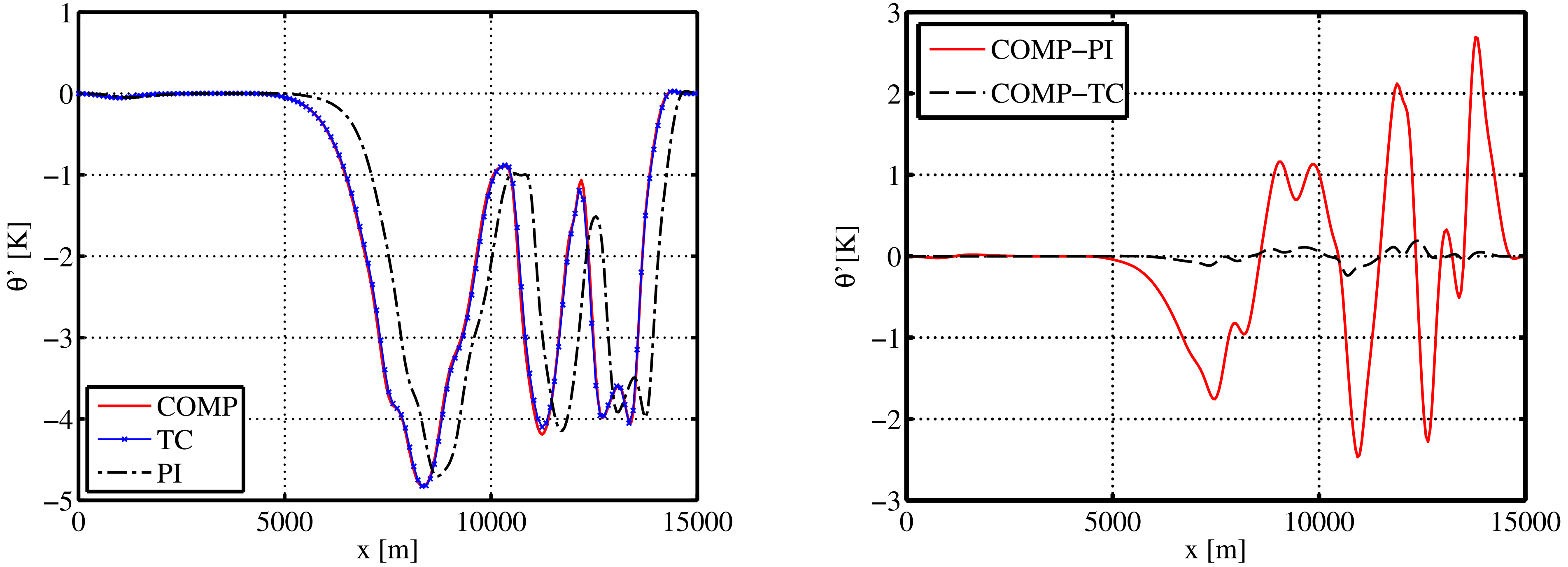
$$\rho^* \theta = \overline{P}\,, \qquad p = \overline{p}(z) + p'\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)$$

* K., Pauluis, JAS, **69**, 961–968, (2012)

Straka's test



Straka's test – model comparison



Limit regimes in atmospheric flows

Sound-proof limits

Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State

Vater, K., Knio, Acta Geophysica, **59**, 120–147, (2011)

Scale-dependent time integration

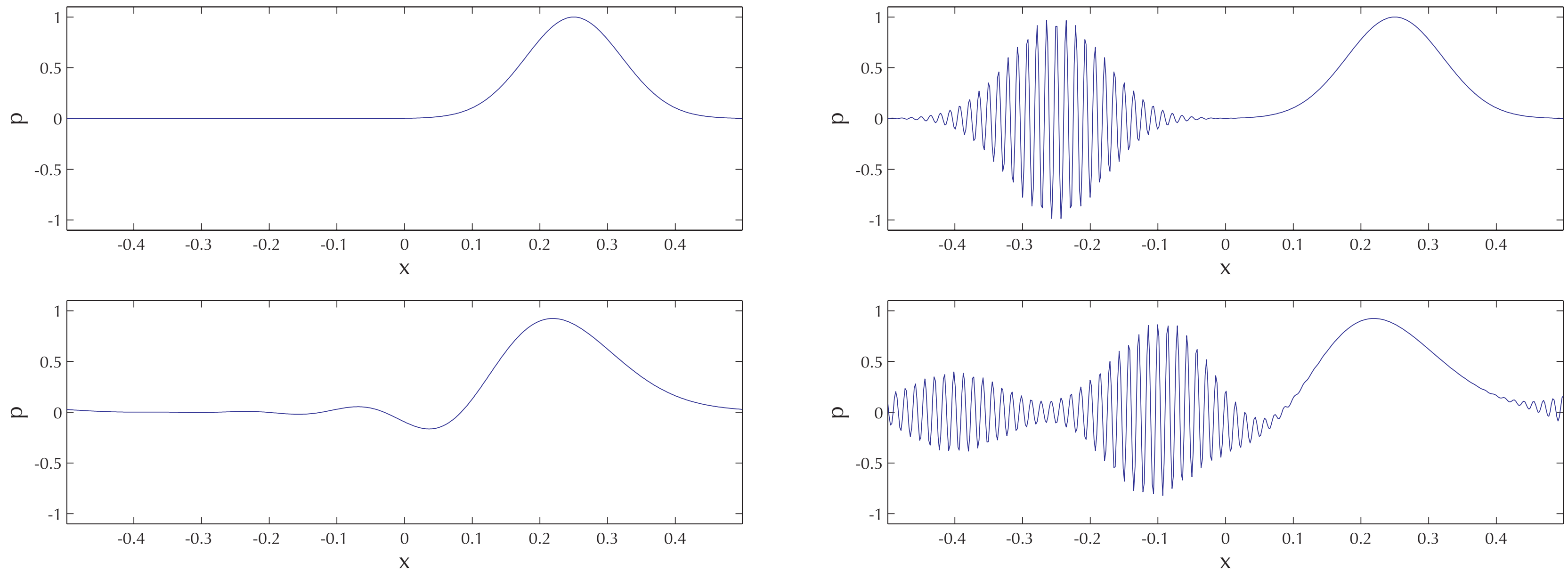
Why not simply solve the full compressible equations?

Competing approaches:	**model codes**
• Split-explicit / multi-rate methods, e.g.,	
– Runge-Kutta (slow) + forward-backward (fast), e.g., *Wicker & Skamarock, MWR, (98), ... ;*	*MM5, LM, WRF ...*
– Multirate infinitesimal schemes, peer methods *Wensch et al., BIT, (09);*	*ASAM, ...*
• Semi-implicit / linearly implicit schemes	
– explicit advection, damped 2nd or 1st-order schemes for fast modes, e.g., *Robert, Japan Met. J., (69), ... ;*	*UKMO, ...*
– linearly implicit Rosenbrock-type methods, e.g., *Reisner et al., MWR, (05), ...;*	*ASAM, LANL Hurricane model, ...*
• Fully implicit integration	

Scale-dependent time integration

Why not simply solve the full compressible equations?

Linear acoustics, simple wave initial data, periodic domain
(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***



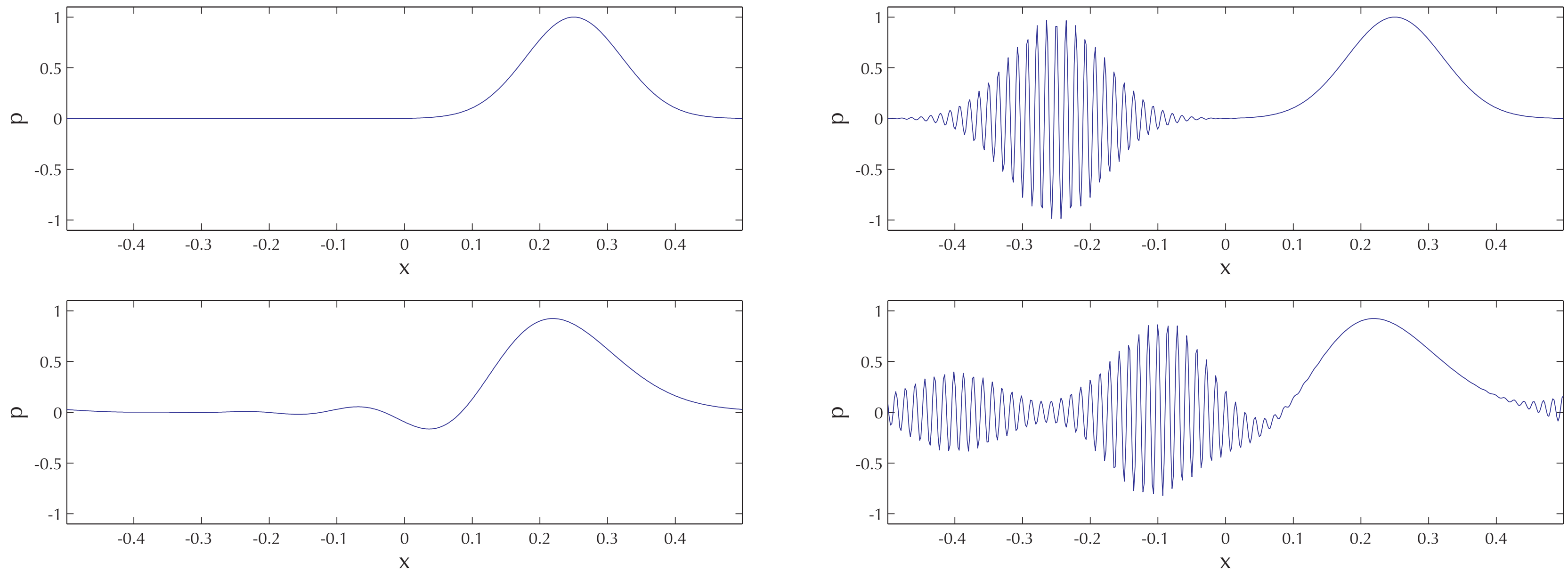
$t = 0$

$t = 3$

Scale-dependent time integration

Why not simply solve the full compressible equations?

Linear acoustics, simple wave initial data, periodic domain
(integration: implicit midpoint rule, staggered grid, 512 *grid pts.,* $\mathrm{CFL} = 10$*)*



$t = 0$

$t = 3$

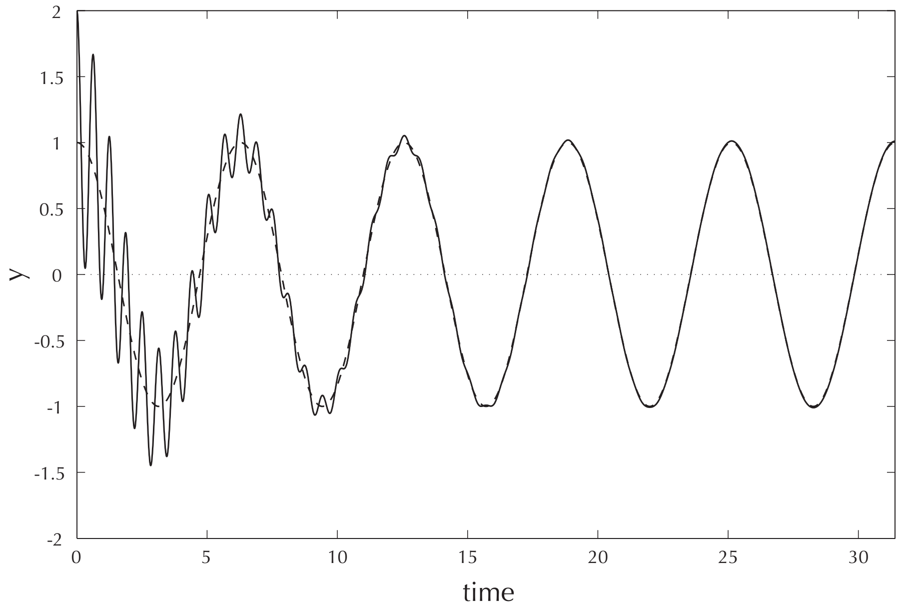
Ideas:

- Slave short waves $(c\Delta t/\ell > 1)$ to long waves $(c\Delta t/\ell \leq 1)$
- with pseudo-incompressible limit behavior

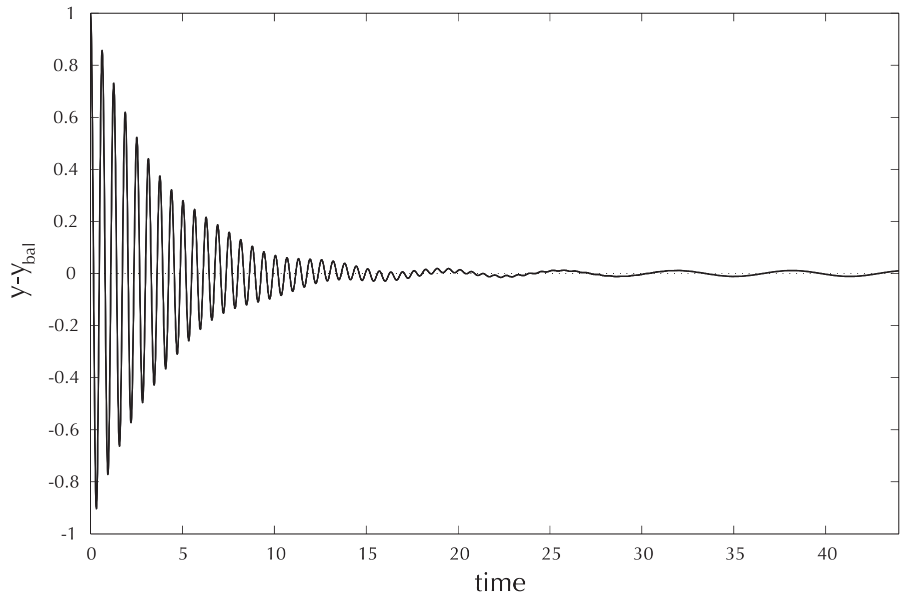
"super-implicit" scheme
non-standard multi grid
projection method

Scale-dependent time integration

$$\varepsilon \ddot{y} + \varepsilon \kappa \dot{y} + y = \cos(t)\,, \qquad \begin{cases} y(0) = 1 + a \\ \dot{y}(0) = 0 \end{cases}, \qquad (\varepsilon = 0.01)$$



$y(t)$



$y(t) - \cos(t)$

Scale-dependent time integration

$$\varepsilon\ddot{y} + \varepsilon\kappa\dot{y} + y = \cos(t)$$

Slow-time asymptotics for $\varepsilon \ll 1$:

$$y(t) = y^{(0)}(t) + \varepsilon y^{(1)}(t) + \dots, \qquad \begin{aligned} y^{(0)}(t) &= \cos(t) \\ y^{(1)}(t) &= -(\ddot{y}^{(0)} + \kappa\dot{y}^{(0)})(t) \end{aligned}$$

Associated "super-implicit" discretization *(extreme BDF)*:

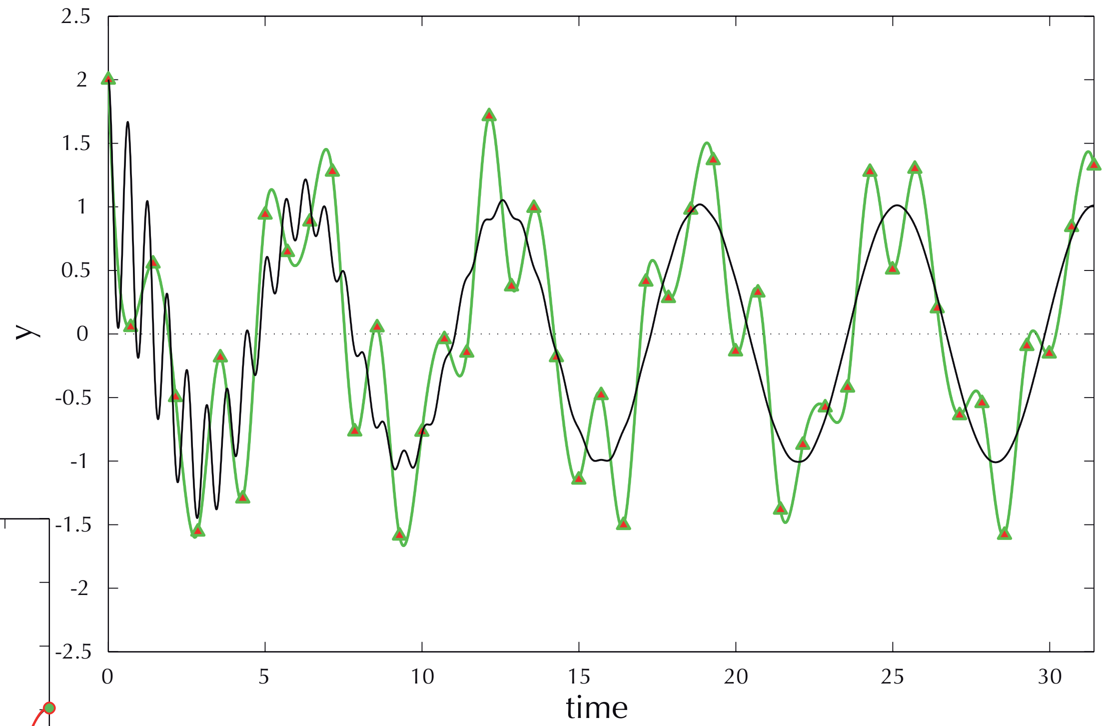
$$\begin{aligned} y^{n+1} &= \cos(t^{n+1}) - \varepsilon\left[(\delta_t + \kappa)\ \dot{y}\right]^{*,n+1} \\ \dot{y}^{n+1} &= \frac{1}{\Delta t}\left(y^{n+1} - y^n + \frac{1}{2}\left(y^{n+1} - 2y^n + y^{n-1}\right)\right) \end{aligned}$$

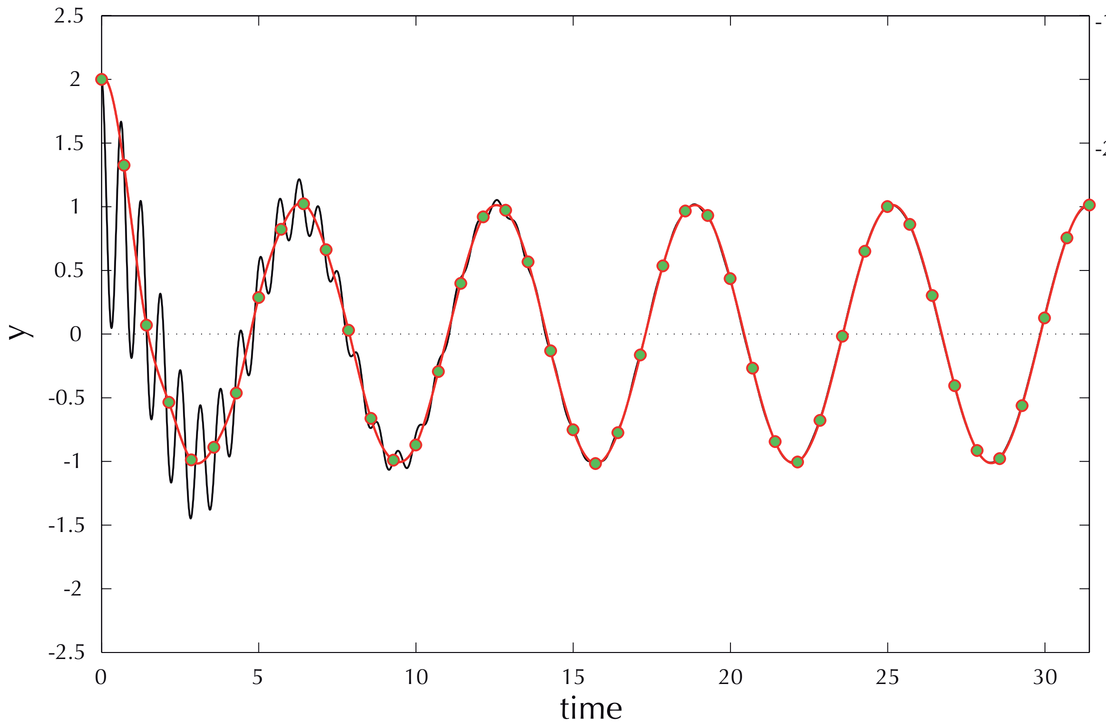
where

$$\begin{aligned} u^{*,n+1} &= 2u^n - u^{n-1} \\ (\delta_t u)^{*,n+1} &= \frac{1}{\Delta t}\left(u^n - u^{n-1} + \frac{3}{2}(u^n - 2u^{n-1} + u^{n-2})\right) \end{aligned}$$

Scale-dependent time integration

Implicit midpoint rule $\Delta t = 7\sqrt{\varepsilon}$

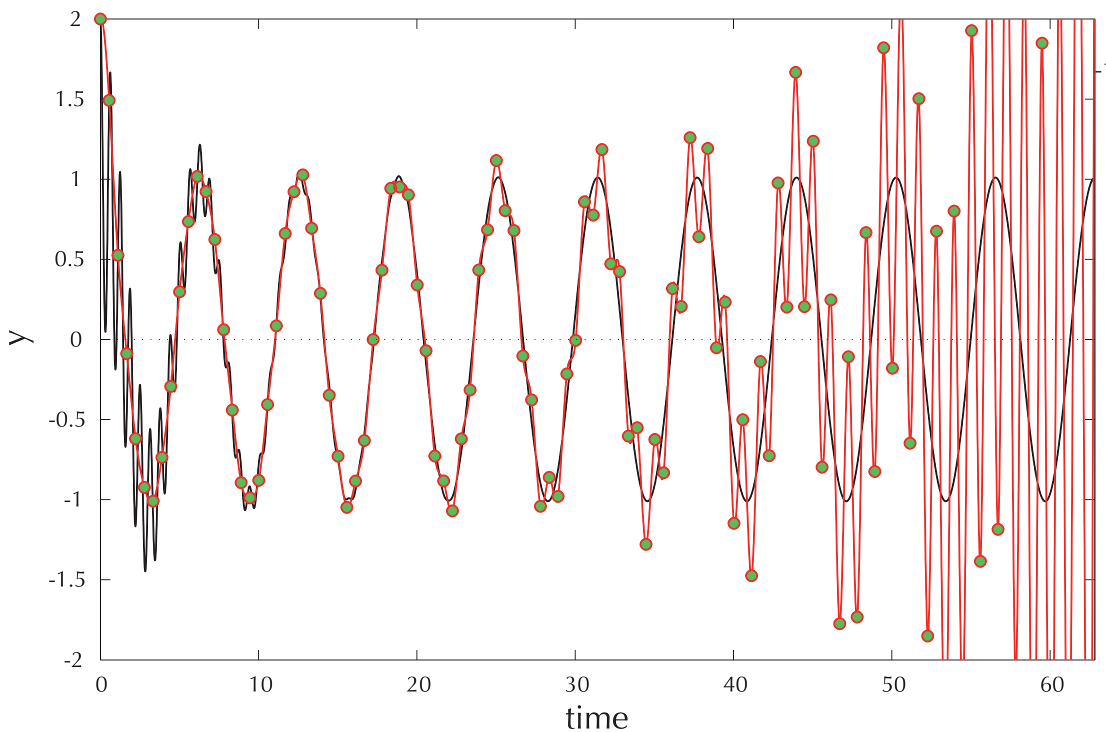


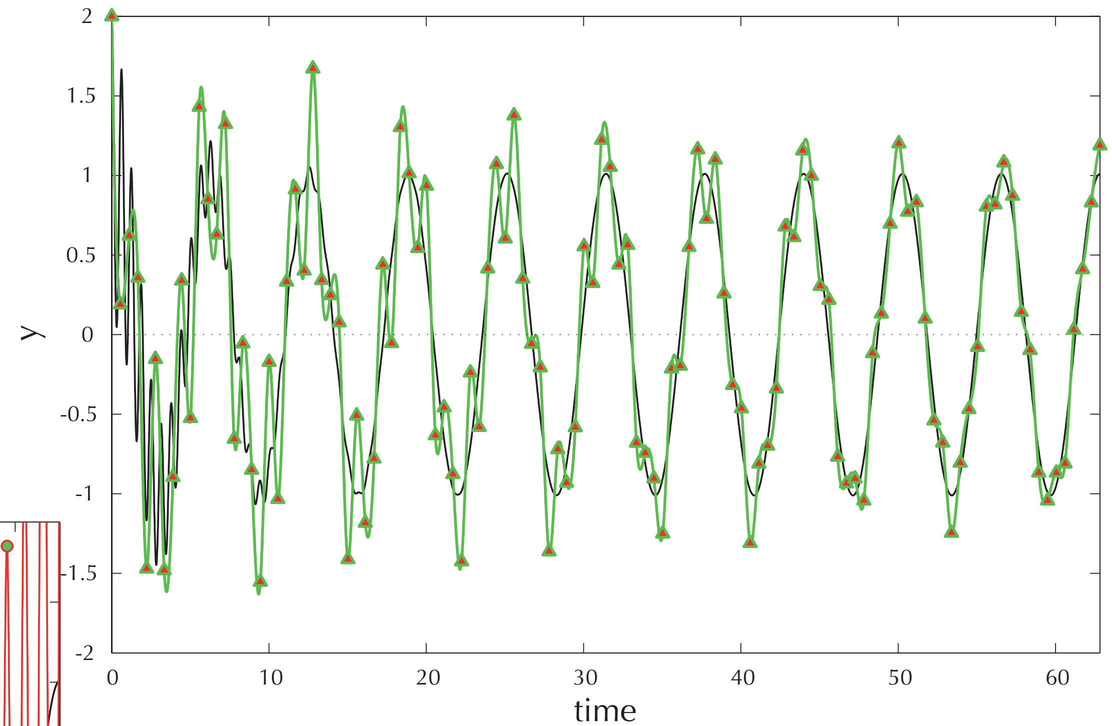


$\Delta t = 7\sqrt{\varepsilon}$ Super-implicit scheme

Scale-dependent time integration

Implicit midpoint rule $\Delta t = 5.55\sqrt{\varepsilon}$



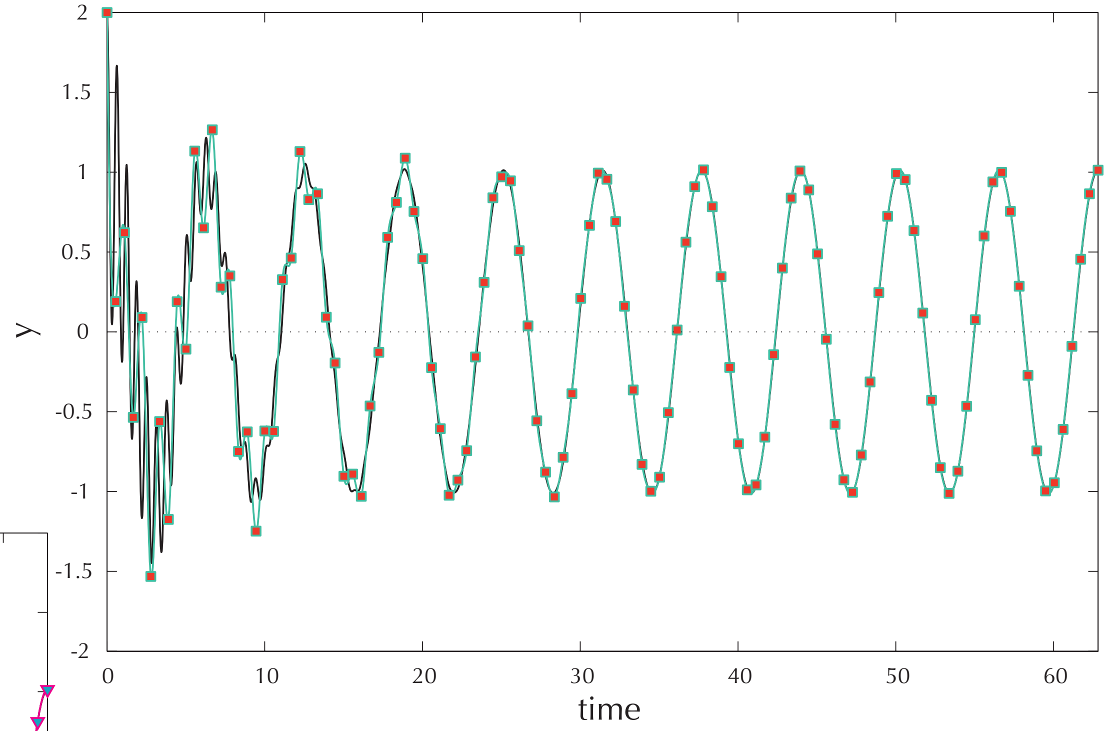


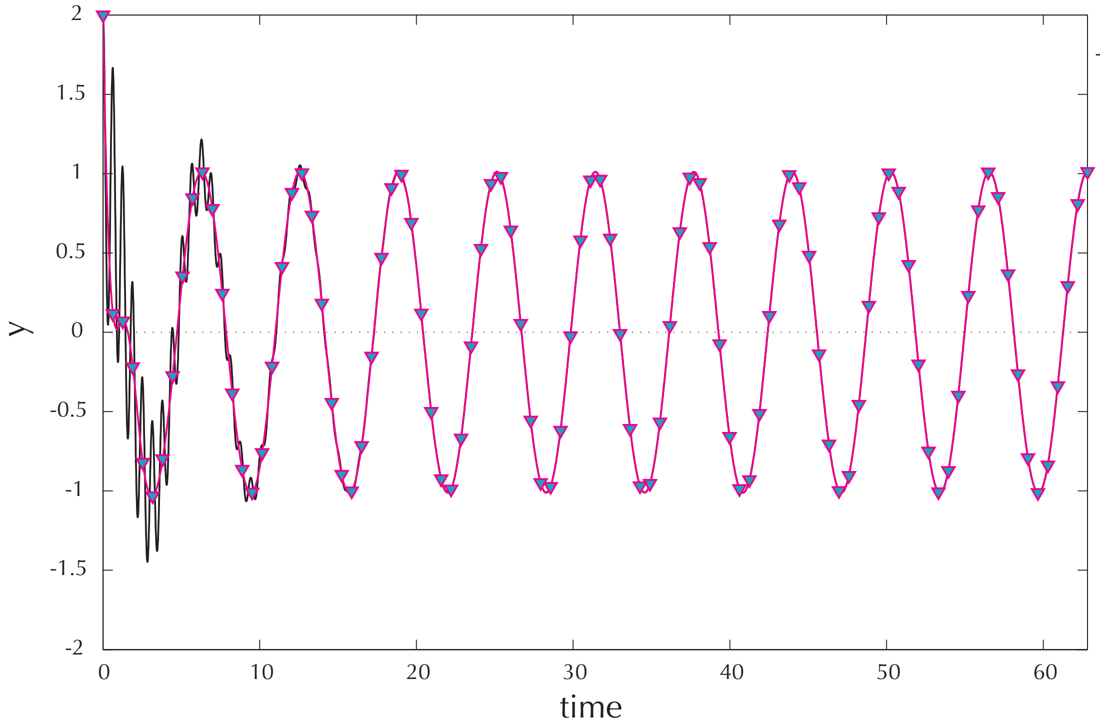
$\Delta t = 5.55\sqrt{\varepsilon}$ Super-implicit scheme

Scale-dependent time integration

Blended scheme $\Delta t = 5.55\sqrt{\varepsilon}$

$$\Delta y\big|_{\mathrm{BL}} = \eta\,\Delta y\big|_{\mathrm{IMP}} + (1-\eta)\,\Delta y\big|_{\mathrm{SupI}}$$





$\Delta t = 5.55\sqrt{\varepsilon}$ BDF2 – for comparison

Scale-dependent time integration

Compressible flow equations:

$$\begin{aligned} \boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\ (\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \pi &= -\rho g \boldsymbol{k} \\ \boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0 \end{aligned}$$

$$P = p^{\frac{1}{\gamma}} = \rho \theta \,, \qquad \pi = p / \Gamma P \,, \qquad \Gamma = c_p / R$$

Scale-dependent time integration

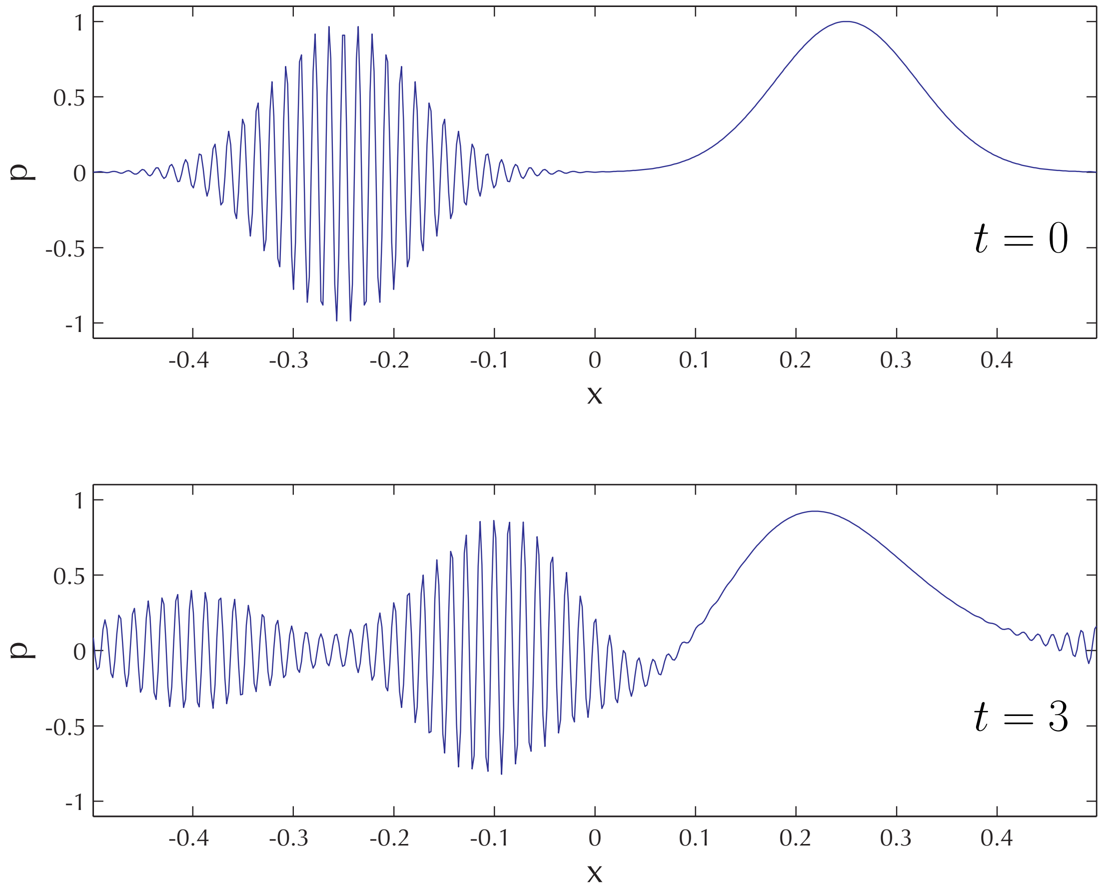
For starters: 1D Linear acoustics:

$$u_t + p_x = 0$$

$$p_t + c^2 u_x = 0$$

Desired:

- remove underresolved modes
- minimize dispersion for marginally resolved modes

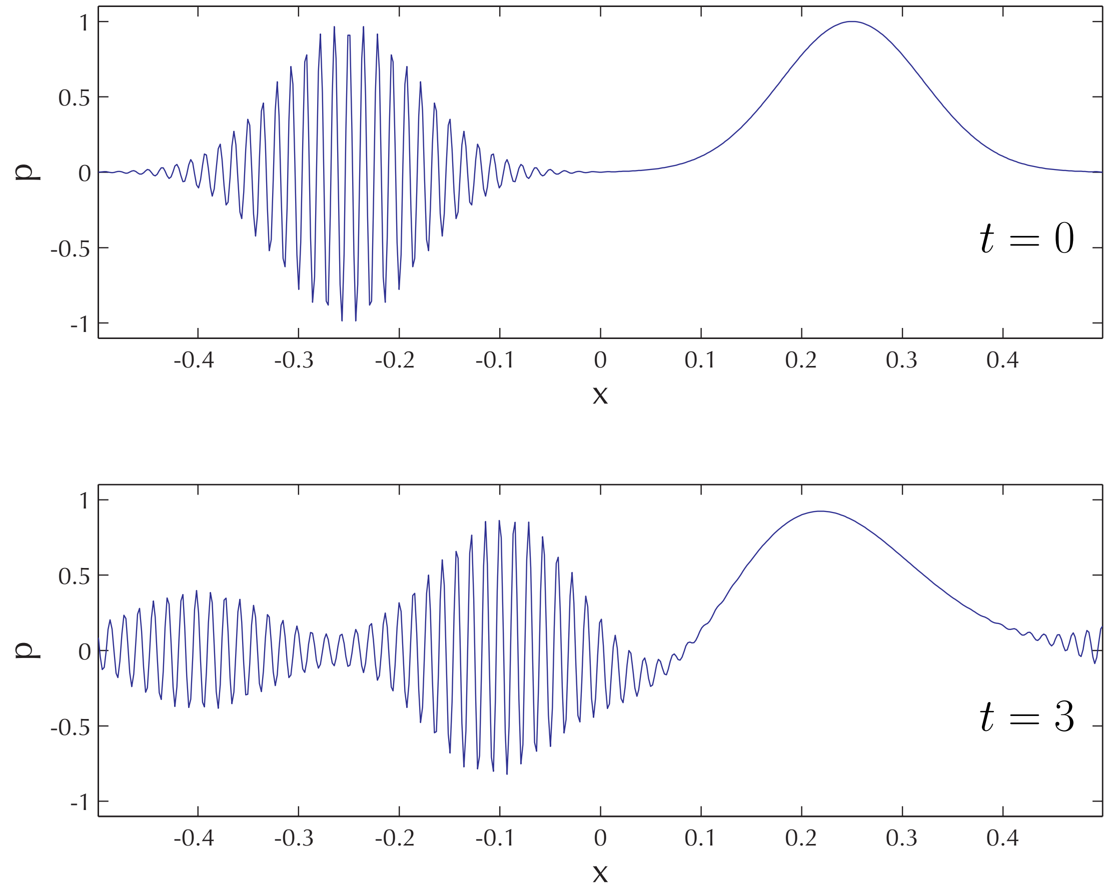


Scale-dependent time integration

1D Linear acoustics:

$$u_t + p_x = 0$$

$$p_t + c^2 u_x = 0$$



Desired:

- remove underresolved modes
- minimize dispersion for marginally resolved modes

Strategy:

scale-dependent IMP-SupI-Blended scheme via **multi grid**

Scale-dependent time integration

Implicit mid-point rule for linear acoustics

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial}{\partial x} p^{n+\frac{1}{2}} = 0\,, \qquad \frac{p^{n+1} - p^n}{\Delta t} + c^2 \frac{\partial}{\partial x} u^{n+\frac{1}{2}} = 0$$

with

$$X^{n+\frac{1}{2}} = \frac{1}{2}\left(X^{n+1} + X^n\right)$$

Implicit problem for half-time fluxes

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}\,, \qquad p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}}$$

Eliminate $u^{n+\frac{1}{2}}$

$$\left(1 - \frac{c^2 \Delta t^2}{4} \frac{\partial^2}{\partial x^2}\right) p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^n$$

Scale-dependent time integration

Implicit mid-point rule $\Rightarrow$ **super-implicit**

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$\underline{p^{n+\frac{1}{2}}} = \underline{p^n} - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}}$$

key step:

$$\begin{aligned} u^{n+\frac{1}{2}} &= u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}} \\ &= \quad - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}} - \frac{\Delta t}{2}\left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}} \end{aligned}$$

Pressure **"projection"** equation

$$\frac{c^2\Delta t}{2}\frac{\partial^2}{\partial x^2}p^{n+\frac{1}{2}} = c^2\frac{\partial}{\partial x}u^n + \left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}}$$

Scale-dependent time integration

Scale-dependence via **multi-grid**

$$p = \sum_{j=1}^{J} p^{(j)}$$

where

$$p^{(j)} = (1 - P \circ R)\ R^{j-1} p \qquad \text{with} \qquad \begin{array}{lcl} R & : & \text{MG restriction} \\ P & : & \text{MG prolongation} \end{array}$$

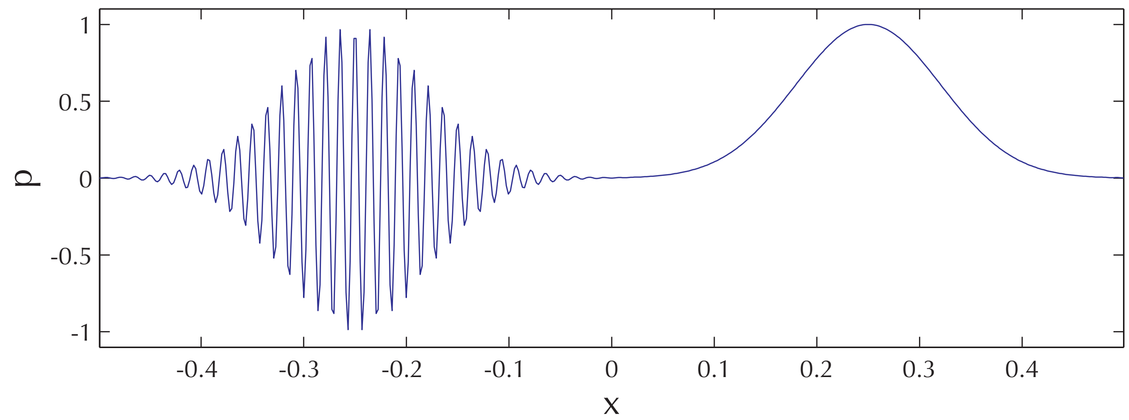
scale-dependent blending

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}$$

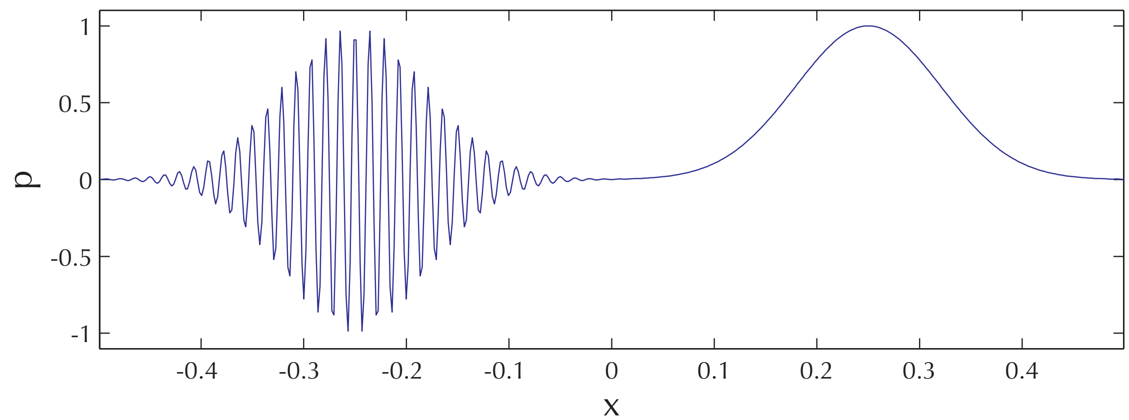
$$\sum_j \boldsymbol{\eta}^{(j)} {p^{(j)}}^{n+\frac{1}{2}} = \sum_j \boldsymbol{\eta}^{(j)} {p^{(j)}}^{n} - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}} - \sum_j (1 - \boldsymbol{\eta}^{(j)}) \frac{\Delta t}{2} \left(\frac{\partial p^{(j)}}{\partial t} \right)^{\mathbf{BD}, n+\frac{1}{2}}$$

Scale-dependent time integration

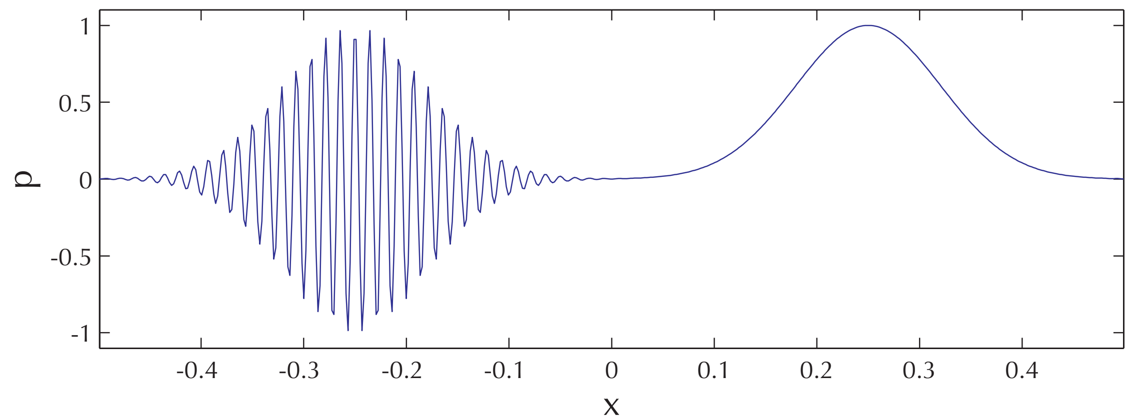
implicit midpoint



new scheme



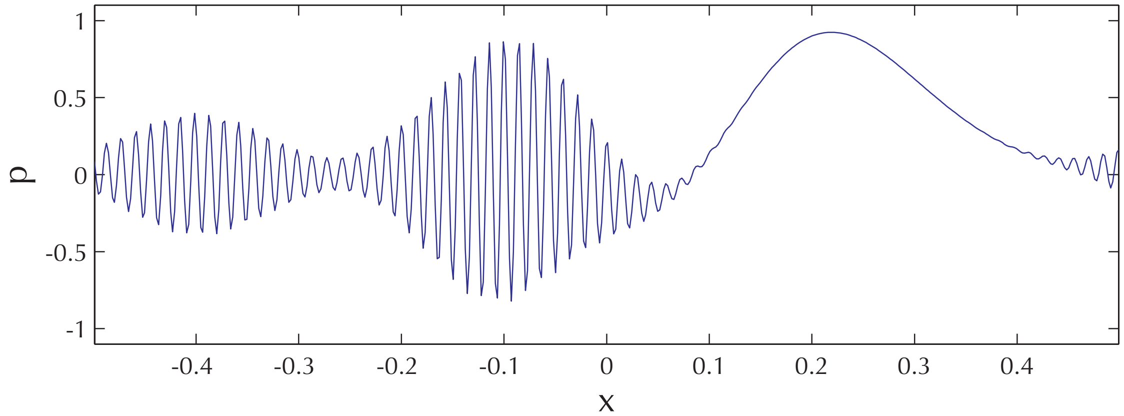
BDF2



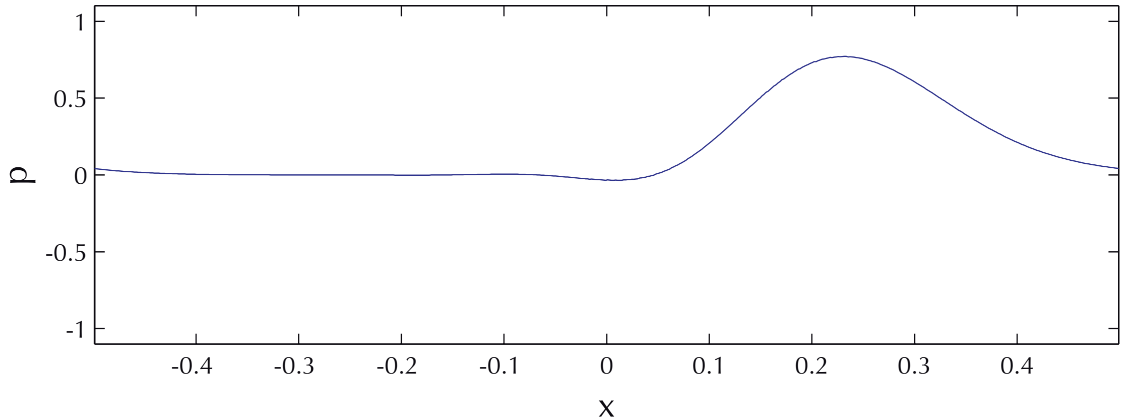
$t = 0$

Scale-dependent time integration

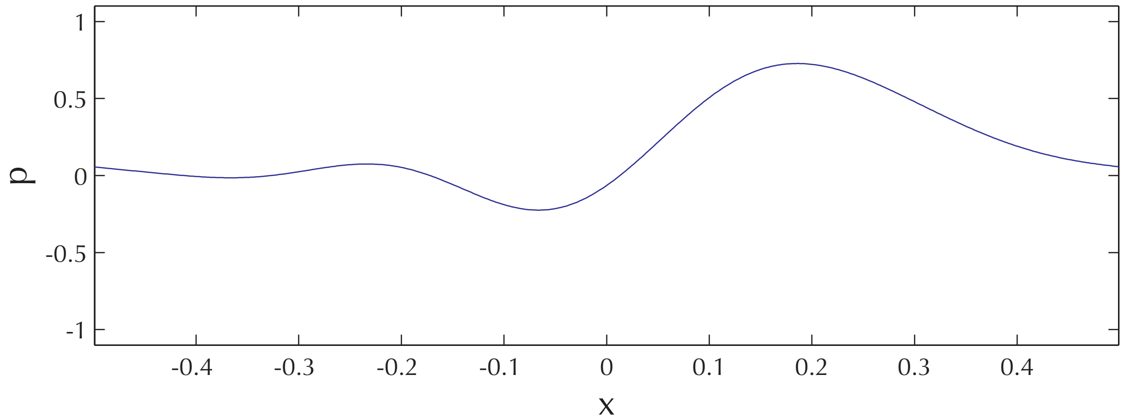
implicit midpoint



new scheme



BDF2

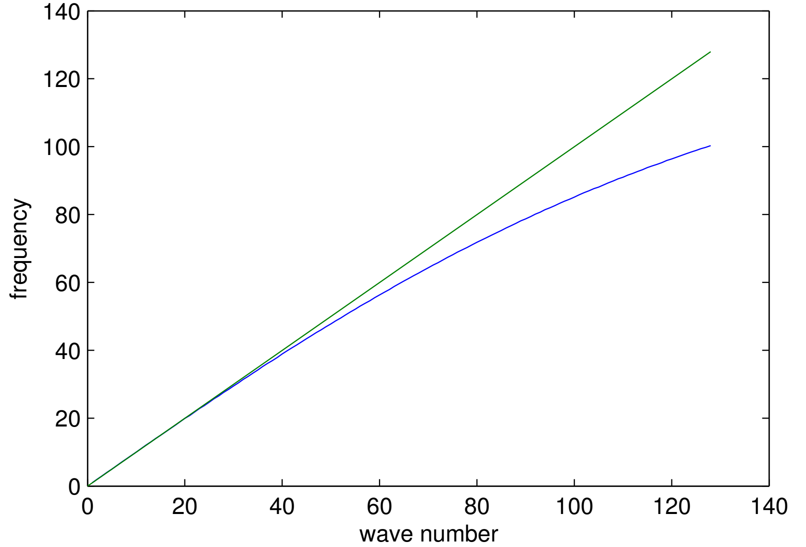


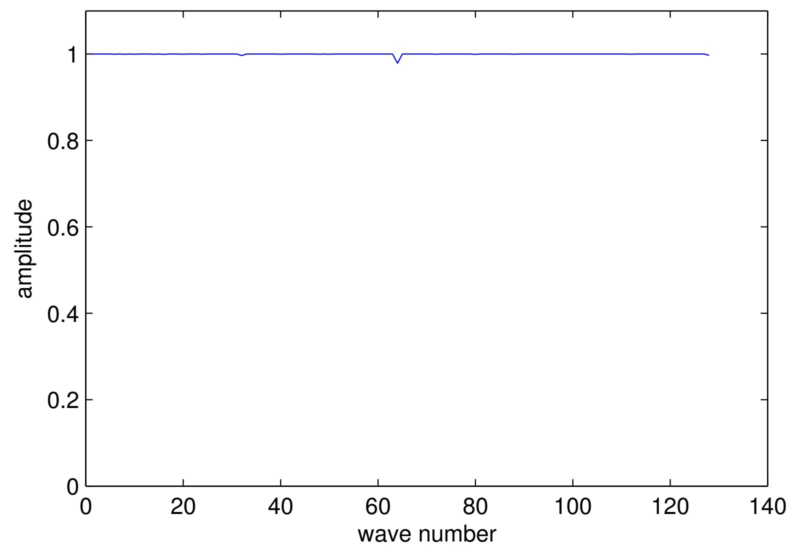
$t = 3$

Model Equations – Dispersion Relation and Amplitude

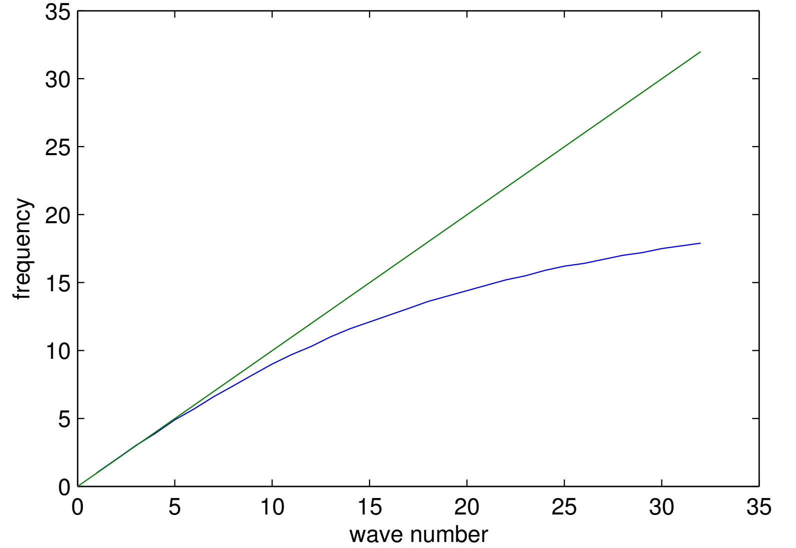
Implicit midpoint rule:

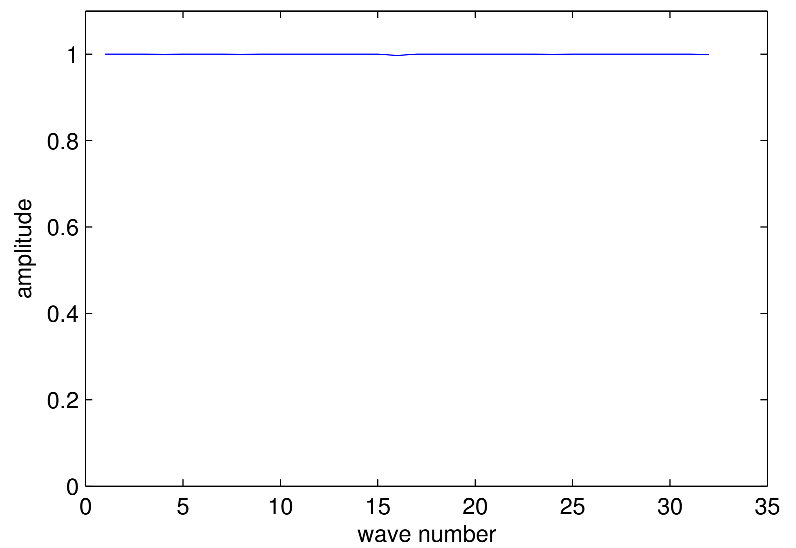
CFL=1





CFL=10

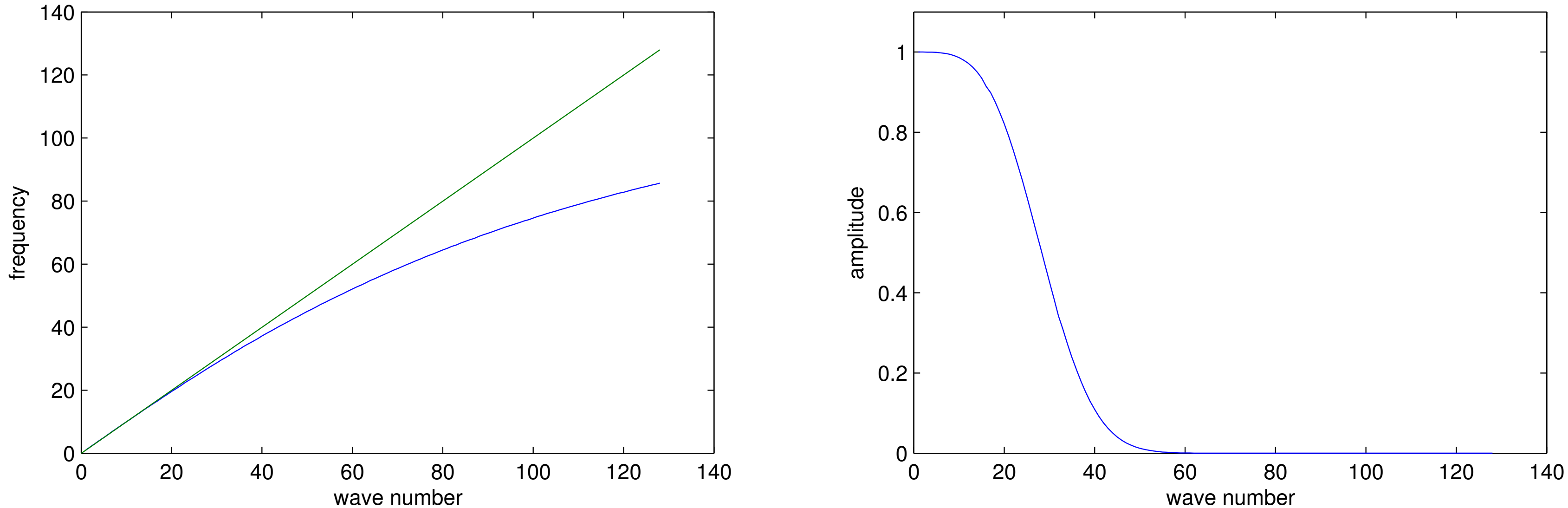




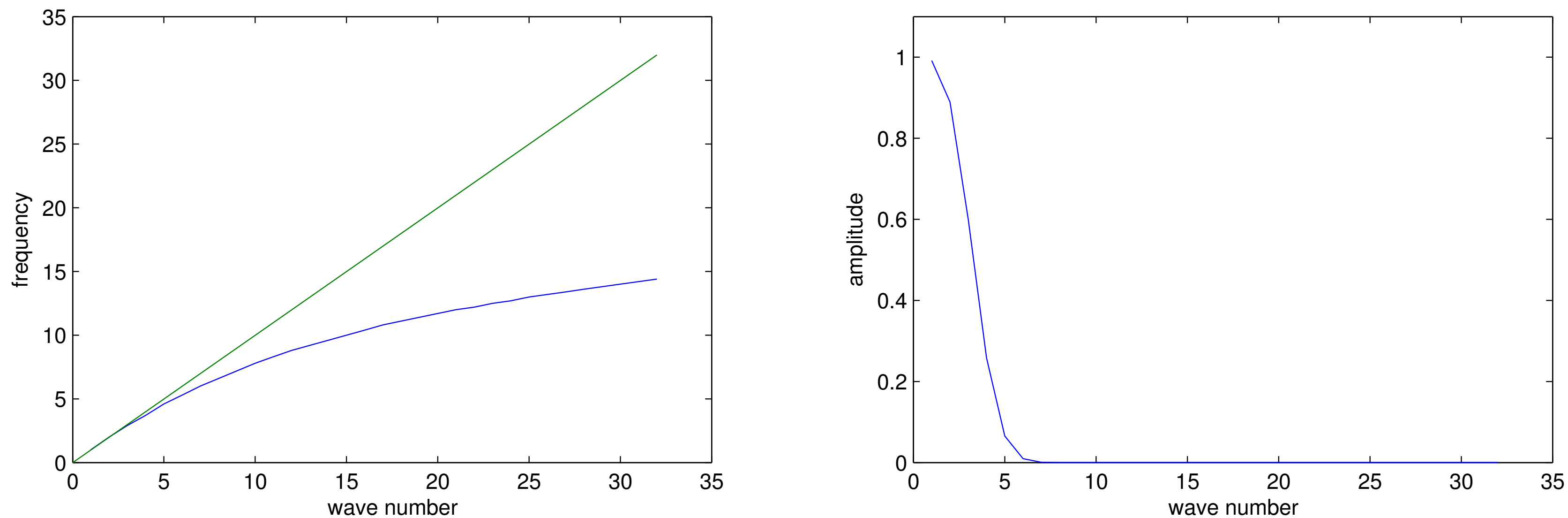
Model Equations – Dispersion Relation and Amplitude

BDF-2:

CFL=1



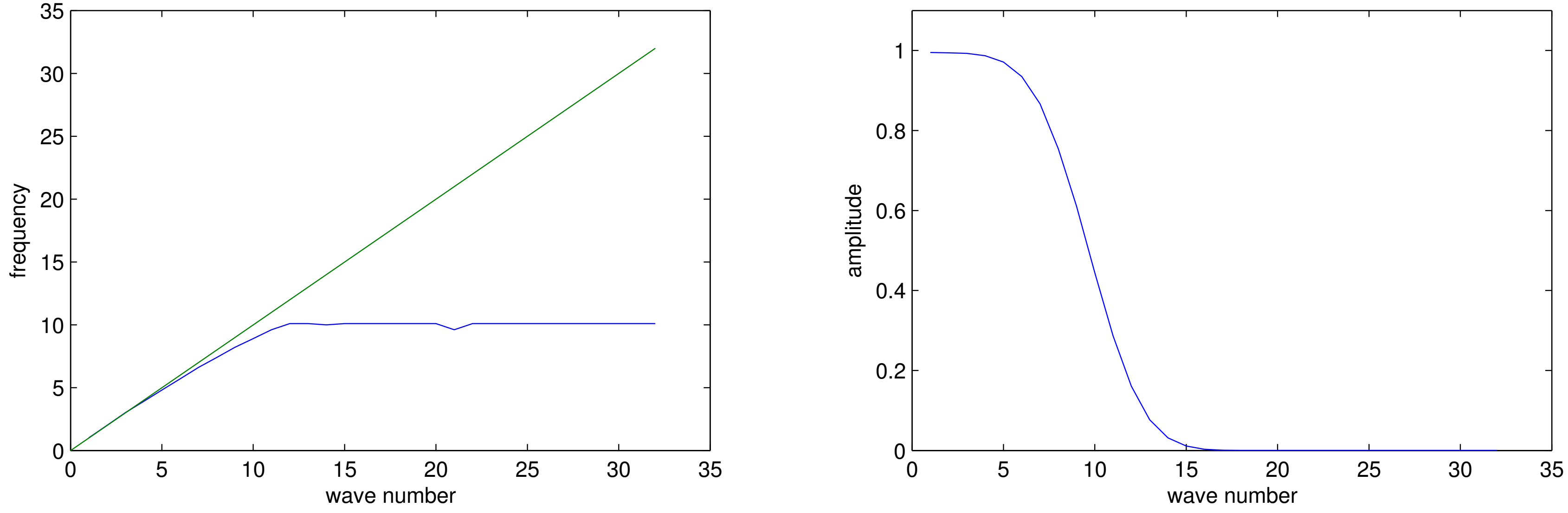
CFL=10



Dispersion Relation and Amplitude

Blended Scheme

CFL=10



Limit regimes in atmospheric flows

Sound-proof limits

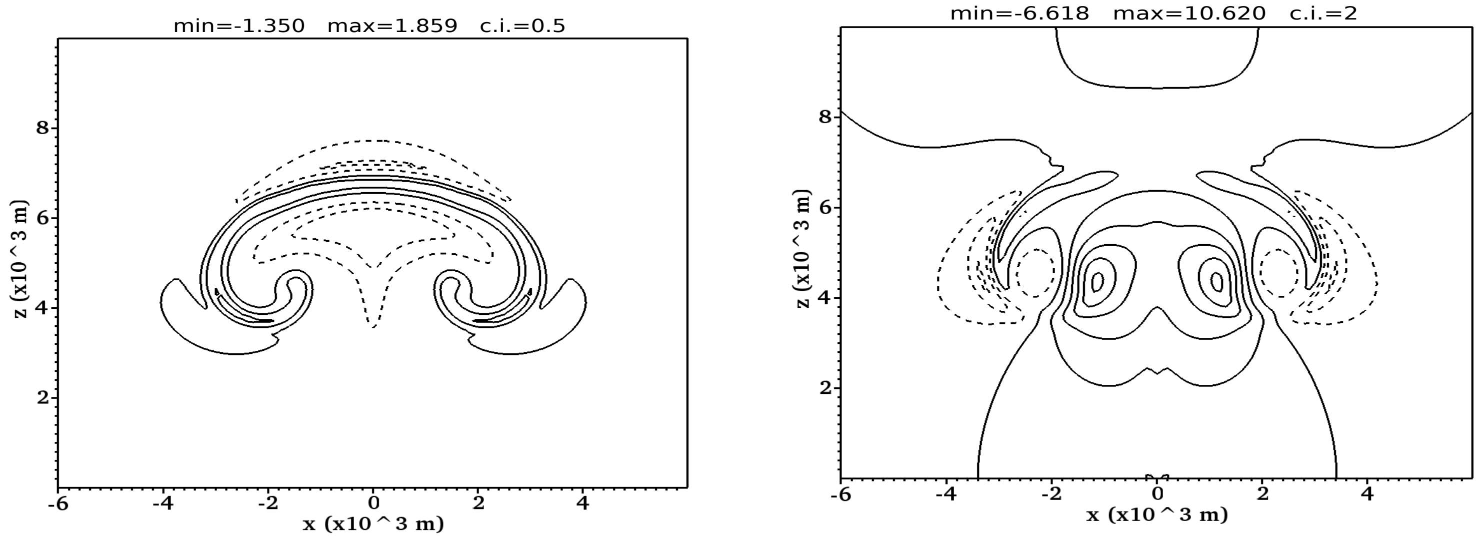
Semi-implicit scheme for compressible flows

Scale-dependent time integration

Extensions: Moisture & general Eqs. of State

Moist pseudo-incompressible model

Bryan’s moist bubble test case



Run with straight pseudo-incompressible model*

Thermodynamically consistent version is work in progress

* O’Neill, K., Atmos. Res., accepted, (2013); see also: Almgren et al. , Astrophys. J., **684**, 449–470, (2008)

Conclusions

Publications

[1] Klein R., *Asymptotics, structure, and integration of sound-proof atmospheric flow equations*, Theor. & Comput. Fluid Dyn., **23**, 161-195, (2009)

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