TECHNICAL MEMORANDUM

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Treatment of the Coriolis terms in semi Lagrangian spectral models

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ABSTRACT

In a conventional three-time-level semi-Lagrangian scheme, the Coriolis terms are treated in an explicit centred fashion. This option is not available in a two-time-level scheme, and an alternative treatment must be sought. Two possible alternatives are tested here in the framework of a three-time-level scheme. Both are stable and accurate, but only one generalizes easily to a rotated coordinate system.

1. INTRODUCTION

In September 1991, a new high-resolution (T213, 31-level) spectral model became operational at the European Centre for Medium-Range Weather Forecasts (ECMWF). A considerable gain in the computational efficiency of the model was required to produce operational forecasts at this resolution with the available computer resources, and to this end the new model used a three-time-level semi-Lagrangian semi-implicit integration scheme as pioneered by *Robert* (1981, 1982). The formulation of this model and details of its performance are described in a recent paper by *Ritchie et al* (1995).

In March 1994 a new version of the model was implemented operationally. While the scientific details of the forecast model itself remained essentially the same, the new code included many additional features required for three- and four-dimensional variational data assimilation (*Thépaut and Courtier*, 1991; *Andersson et al*, 1994) and for determining optimal unstable perturbations for ensemble prediction (*Buizza et al*, 1993). This new code was developed jointly by ECMWF (where it is known as the Integrated Forecast System, IFS) and Météo-France (where it is known as ARPEGE). In the present context, one aspect of note is the operational use of this model by Météo-France in rotated and stretched mode (*Courtier and Geleyn*, 1988).

The model is currently being adapted to make use of a *two*-time-level semi-Lagrangian scheme, both to take advantage of greater computational efficiency (*Staniforth and Côté*, 1991) and in the hope of reducing storage requirements when the model is used for four-dimensional data assimilation. Since the Coriolis terms can then no longer be treated in a simple explicit manner as in *Ritchie et al* (1995), it is necessary to seek an alternative treatment. There are at least two possibilities, both of which can be evaluated within the framework of the well-established three-time-level scheme. These investigations form the subject of the present paper.

2. SEMI-IMPLICIT TREATMENT OF CORIOLIS TERMS

The first and more obvious alternative is to treat the Coriolis terms semi-implicitly (i.e., to average them in time and space along the trajectory). As in *Ritchie et al* (1995), the resulting set of coupled equations to be solved for the variables at the new time level $(t+\Delta t)$ has the same form whether the underlying scheme is Eulerian or semi-Lagrangian. For simplicity, the details are set out here for the corresponding shallow-water equation model. The equations take the form:

$$U^+ - f\Delta t V^+ + \frac{\Delta t}{a}\frac{\partial \phi^+}{\partial \lambda} = R_1 \tag{2.1}$$

$$V^+ + f\Delta t U^+ + \frac{\Delta t}{a}\cos\theta\frac{\partial \phi^+}{\partial \theta} = R_2 \tag{2.2}$$

$$\phi^+ + \Phi\Delta t D^+ = S_3 \tag{2.3}$$

where the superscript + indicates unknown values at $(t+\Delta t)$, while the right-hand sides contain all the known terms. In (2.1)-(2.3) U and V are the wind components multiplied by $\cos\theta$, ϕ is geopotential, D is divergence, and Φ is the mean geopotential.

Transforming (2.1)-(2.3) to spectral space and taking the curl and divergence of the equations for the wind components yields, for each zonal wavenumber m and for $m \le n \le N$ where N is the maximum total wavenumber:

$$(1-i\alpha_n^m)\zeta_n^+ + 2\Omega\Delta t\left[\frac{(n+1)}{n}\varepsilon_n^m D_{n-1}^+ + \frac{n}{(n+1)}\varepsilon_{n+1}^m D_{n+1}^+\right] = (S_1)_n \tag{2.4}$$

$$(1-i\alpha_n^m)D_n^+ - 2\Omega\Delta t\left[\frac{(n+1)}{n}\varepsilon_n^m \zeta_{n-1}^+ + \frac{n}{(n+1)}\varepsilon_{n+1}^m \zeta_{n+1}^+\right]$$

$$-\frac{\Delta t n(n+1)}{a^2}\phi_n = (S_2)_n \tag{2.5}$$

$$\phi_n^+ + \Phi\Delta t D_n^+ = (S_3)_n \tag{2.6}$$

where

$$\alpha_n^m = \frac{2\Omega m\Delta t}{n(n+1)}, \quad \varepsilon_n^m = \left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2},$$

$$S_1 = curl(R_1,R_2) \text{ and } S_2 = div(R_1,R_2).$$

In (2.4)-(2.6) the zonal wavenumber index m has for clarity been omitted from the spectral coefficients ζ_n^+, D_n^+, ϕ_n^+. Equations (2.4)-(2.6) are derived from (2.1)-(2.3) using standard recurrence relations and orthogonality properties of the Legendre polynomials, and in practice may be obtained by adapting *Bourke's* (1972) derivation of the spectral shallow-water equations.

These equations differ in two important ways from those obtained without the additional Coriolis terms. First, the vorticity and divergence equations are coupled together. Second, although an independent set of equations is obtained for each zonal wavenumber m, the vorticity components at total wavenumber n are coupled to the divergence components at total wavenumbers $(n-1)$ and $(n+1)$, and vice versa.

To solve the equations, all the spectral components $(m \le n \le N)$ for each variable are assembled into a vector for each wavenumber m. Thus, (2.4)-(2.6) become:

$$\underline{\underline{J}}\,\underset{\sim}{\zeta}^{+} + \underline{\underline{F}}\,\underset{\sim}{D}^{+} = \underset{\sim}{S}_1 \tag{2.7}$$

$$\underline{\underline{J}}\,\underset{\sim}{D}^{+} - \underline{\underline{F}}\,\underset{\sim}{\zeta}^{+} - \underline{\underline{L}}\,\underset{\sim}{\phi}^{+} = \underset{\sim}{S}_2 \tag{2.8}$$

$$\underset{\sim}{\phi}^{+} + \Phi \Delta t \underset{\sim}{D}^{+} = \underset{\sim}{S}_3 \tag{2.9}$$

where

$$\underline{\underline{J}} = diag\{1 - i\alpha_n^m\},$$

$$\underline{\underline{F}} = 2\Omega\Delta t \begin{bmatrix} 0 & f_m^+ & & & & \\ f_{m+1}^- & 0 & f_{m+1}^+ & & & \\ & f_{m+2}^- & 0 & f_{m+2}^+ & & \\ & & \ddots & \ddots & \ddots & \\ & & & f_{N-1}^- & 0 & f_{N-1}^+ \\ & & & & f_N^- & 0 \end{bmatrix},$$

$$f_n^+ = \frac{n}{(n+1)}\varepsilon_{n+1}^m,$$

$$f_n^- = \frac{(n+1)}{n}\varepsilon_n^m,$$

$$\underline{\underline{L}} = \frac{\Delta t}{a^2} diag\{n(n+1)\}.$$

Equations (2.7)-(2.9) can then be combined to give:

$$(\underline{\underline{J}} + \underline{\underline{F}}\underline{\underline{J}}^{-1}\underline{\underline{F}} + \Phi\Delta t\underline{\underline{L}})\underset{\sim}{D}^{+} = \underset{\sim}{S}_2 + \underline{\underline{F}}\underline{\underline{J}}^{-1}\underset{\sim}{S}_1 + \underline{\underline{L}}\underset{\sim}{S}_3. \tag{2.10}$$

The matrix on the left-hand side of (2.10) is pentadiagonal with zeros on the sub- and superdiagonals; thus

(2.10) decouples into two tridiagonal systems, one for the "even" components and one for the "odd" components. The tridiagonal matrices have *complex* entries, but they are diagonally dominant and the usual solution algorithm (equivalent to Gaussian elimination without pivoting) is stable. Equation (2.10) is essentially the same as that obtained by *Côté and Staniforth* (1988) for a two-time-level semi-Lagrangian spectral model.

Proceeding to the case of a multi-level model, the semi-implicit equations including Coriolis terms may be solved by diagonalizing the vertical operators which couple the unknowns at different model levels, resulting in a set of equations of the form (2.7)-(2.9) for each vertical mode, with the mean geopotential height Φ in (2.9) being replaced by the equivalent geopotential depth for each mode.

The semi-implicit treatment of the Coriolis terms was tested in the three-time-level semi-Lagrangian spectral model, using the "vertically non-interpolating" version (*Ritchie et al*, 1995) at resolution T106, 31 levels, and compared with the standard treatment of the Coriolis terms over a set of six independent cases evenly spaced throughout the year. In terms of verification scores, the choice of treatment of the Coriolis terms has very little impact on the results. A typical example is shown in Figure 1, for the 500 hPa height field over the Northern Hemisphere. Visual comparison of charts confirms that the forecast fields are very similar.

These results suggest that a semi-implicit treatment of the Coriolis terms would also be a viable option in a *two*-time-level semi-Lagrangian integration scheme. A problem would however arise in the case of rotated and stretched coordinates: the rotation would destroy the horizontal separability of the equations to be solved in spectral space (essentially because the Coriolis parameter becomes a function of longitude as well as latitude in the transformed coordinate system). Although it should be possible to solve the resulting set of equations in spectral space, it would certainly be more complicated (and more expensive) than in the non-rotated case.

3. ADVECTIVE TREATMENT OF CORIOLIS TERMS

A less obvious alternative was proposed by *Rochas* (1990). Again for simplicity, we consider the treatment of the Coriolis terms in the shallow-water equations. Recall that semi-Lagrangian schemes on the sphere handle the momentum equations in *vector* form to avoid an instability due to the metric term (*Ritchie*, 1988). Thus, the momentum equation is written as

$$\frac{d\underset{\sim}{v}}{dt} + 2\underset{\sim}{\Omega}\times\underset{\sim}{v} + \underset{\sim}{\nabla}\phi = 0 \qquad (3.1)$$

where the total derivative operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}$$

is discretized in a semi-Lagrangian fashion. In the case of the advection of a *vector* quantity such as $\underset{\sim}{v}=(u,v)$ on the sphere, it has to be borne in mind that the orientation of the local coordinate system changes as we move along the trajectory (Ritchie, 1988).

Rochas (1990) pointed out that, since $\underset{\sim}{v} = d\underset{\sim}{r}/dt$ where $\underset{\sim}{r}$ is the radial position vector, (3.1) can be rewritten as

$$\frac{d}{dt}(\underset{\sim}{v} + 2\underset{\sim}{\Omega}\times\underset{\sim}{r}) + \underset{\sim}{\nabla}\phi = 0. \tag{3.2}$$

In a semi-Lagrangian discretization of (3.2), the Coriolis terms are absorbed into the advection. The change does not affect the trajectory itself, but only the quantity being advected. Since the term $2\underset{\sim}{\Omega}\times\underset{\sim}{r}$ is known everywhere [in component form it is just $(2\Omega a\cos\theta,0)$], the extra term is simply added to $\underset{\sim}{v}$ at the departure point of the trajectory and subtracted again at the arrival gridpoint. The equations to be solved for the variables at the new time-level $(t+\Delta t)$ have exactly the same form as in the case of a simple explicit treatment of the Coriolis terms; the only change is that the right-hand sides have been computed in a different way.

Thus, the idea proposed by *Rochas* (1990) has a clear advantage over the semi-implicit treatment of the Coriolis terms in the rotated and stretched configuration of the model, since the equations resulting from the semi-implicit scheme remain horizontally separable and easy to solve (the Laplacian operator is invariant with respect to a rotation of the coordinate system).

Preliminary tests of this option within the three-time-level scheme, using the "old" (pre-IFS) ECMWF model at T106 resolution, were disappointing. Figure 2 shows the verification scores for the 500hPa height field over the Southern Hemisphere, averaged over a sample of four independent cases. The solid line is for the standard semi-Lagrangian scheme, while the dashed line is for the alternative treatment and shows a clear degradation of the results. It was then realized that in order to speed up the calculation, the treatment of spherical geometry in the determination of the trajectory and handling of the advection of the wind vector (*Ritchie*, 1988) had been replaced by approximations as described by *Ritchie and Beaudoin* (1994), moreover leaving out some higher-order terms. Although these approximations were perfectly adequate for the conventional formulation of the semi-Lagrangian scheme, it was suspected that the accurate treatment of spherical geometry might be more important for the new formulation. This was confirmed by removing the approximations and reverting to a more accurate treatment of the geometry, resulting in the dotted line in Fig 2.

The option based on a semi-Lagrangian discretization of (3.2) has more recently been evaluated in the current (IFS) version of the ECMWF model, which retains an accurate treatment of spherical geometry. Figure 3 shows results for the 500hPa height field over the Northern Hemisphere, meaned over six independent cases. These results indicate if anything a slight advantage of the new formulation over the conventional treatment of the Coriolis terms. As a result of these experiments, the new formulation will shortly become the default option in the operational three-time-level model.

The "advective" treatment of the Coriolis terms in a two-time-level semi-Lagrangian scheme is exactly analogous to that in the three-time-level scheme. There is one potential drawback: since the determination of the trajectory in a two-time-level scheme requires a wind field extrapolated forward in time (*Staniforth and Côté*, 1991), it is conceivable that incorporating the Coriolis terms in the advection might be unstable. This possibility is hinted at by the analysis of *Bates et al* (1995) for a two-time-level scheme based on the advection of potential vorticity, though the scheme of *Bates et al* is by no means equivalent to that proposed here. Fortunately, we now have enough experience with a preliminary two-time-level version of the ECMWF semi-Lagrangian spectral model to be sure that the treatment of the Coriolis terms proposed by *Rochas* (1990) does remain stable when the advection is based on a time-extrapolated wind field.

4. CONCLUSIONS

Two possible treatments of the Coriolis terms in a two-time-level semi-Lagrangian model have been tested by modifying the ECMWF three-time-level semi-Lagrangian spectral model. Including the Coriolis terms in the semi-implicit scheme is stable and accurate, but results in a difficult problem to be solved in spectral space when the model is used with a rotated coordinate system. An alternative scheme proposed by *Rochas* (1990), in which the Coriolis terms are incorporated in the semi-Lagrangian advection, is also stable and accurate provided that the spherical geometry of the problem is handled accurately. This alternative scheme has the additional advantage that the problem to be solved in spectral space has just the same simple form in a rotated coordinate system as in the corresponding unrotated case.

REFERENCES

Andersson, E, J Pailleux, J-N Thépaut, J R Eyre, A P McNally, G A Kelly and P Courtier, 1994: Use of cloud-cleared radiances in three/four-dimensional variational data assimilation. Quart J Roy Meteor Soc, **120**, 627-653.

Bates, J R, Y Li, A Brandt, S F McCormick and J Ruge, 1995: A global barotropic primitive equation model based on the semi-Lagrangian advection of potential vorticity, submitted to Quart J Roy Meteor Soc.

Bourke, W, 1972: An efficient, one-level, primitive equation spectral model. Mon Wea Rev, **100**, 683-689.

Buizza, R, J Tribbia, F Molteni and T Palmer, 1993: Computation of optimal unstable structures for a numerical weather prediction model. Tellus, **45A**, 388-407.

Côté, J and A Staniforth, 1988: A two-time-level semi-Lagrangian semi-implicit scheme for spectral models. Mon Wea Rev, **116**, 2003-2012.

Courtier, P and J-F Geleyn, 1988: A global weather prediction model with variable resolution: application to shallow-water equations. Quart J Roy Meteor Soc, **114**, 1321-1346.

Ritchie, H, 1988: Application of the semi-Lagrangian method to a spectral model of the shallow water equations. Mon Wea Rev, **116**, 1587-1598.

Ritchie, H and C Beaudoin, 1994: Approximations and sensitivity experiments with a baroclinic semi-Lagrangian spectral model. Mon Wea Rev, **122**, 2391-2399.

Ritchie, H, C Temperton, A Simmons, M Hortal, T Davies, D Dent and M Hamrud, 1995: Implementation of the semi-Lagrangian method in a high-resolution version of the ECMWF forecast model. Mon Wea Rev, **123**, 489-514.

Robert, A, 1981: A stable numerical integration scheme for the primitive meteorological equations. Atmos-Ocean, **19**, 35-46.

Robert, A, 1982: A semi-Lagrangian and semi-implicit numerical integration scheme for the primitive meteorological equations. J Meteor Soc Japan, **60**, 319-325.

Rochas, M, 1990: ARPEGE Documentation, Part 2, Ch 6 (available from Météo-France).

Staniforth, A and J Côté, 1991: Semi-Lagrangian integration schemes for atmospheric models - a review. Mon Wea Rev, **119**, 2206-2223.

Thépaut, J-N and P Courtier, 1991: Four-dimensional variational data assimilation using the adjoint of a multilevel primitive-equation model. Quart J Roy Meteor Soc, **117**, 1225-1254.

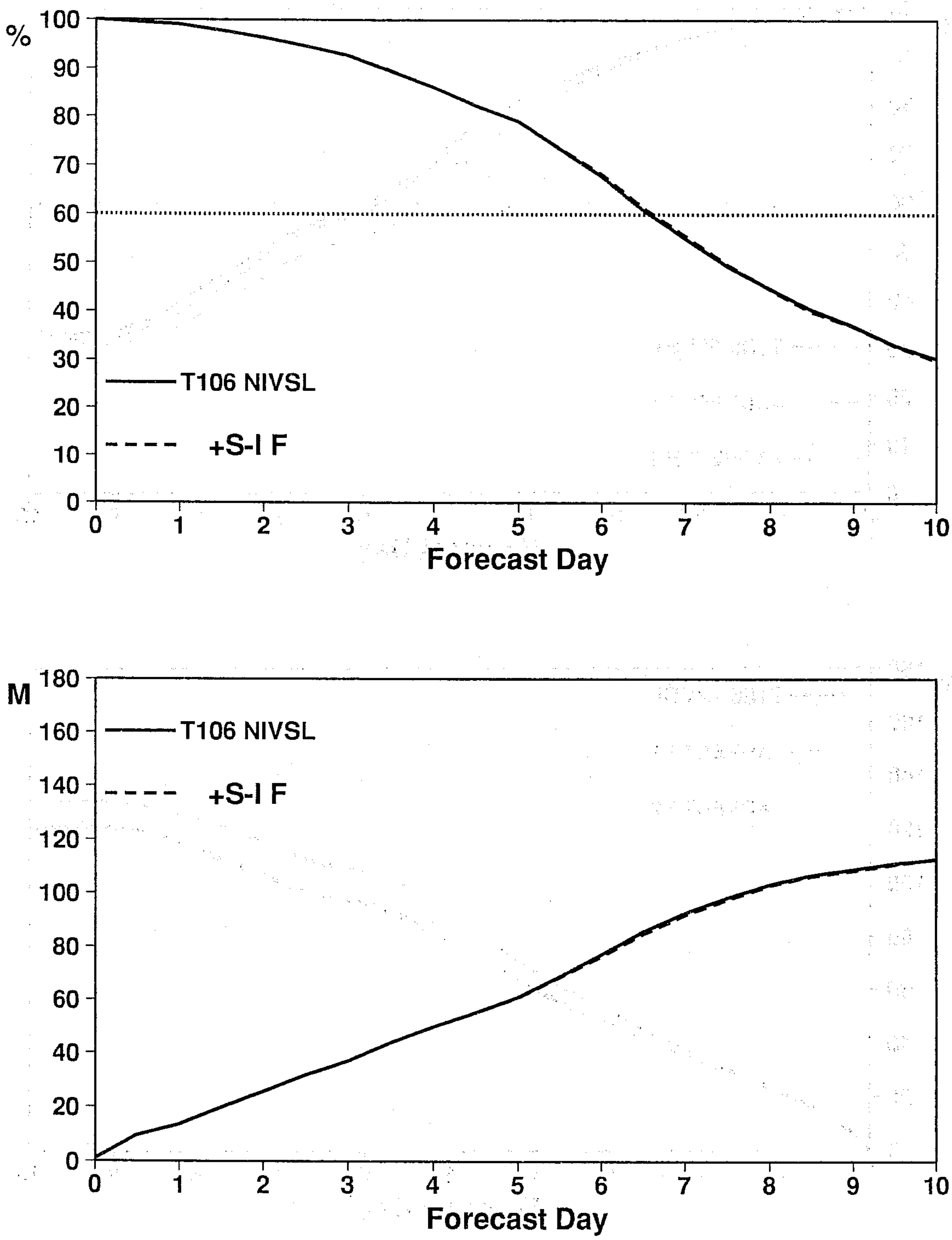
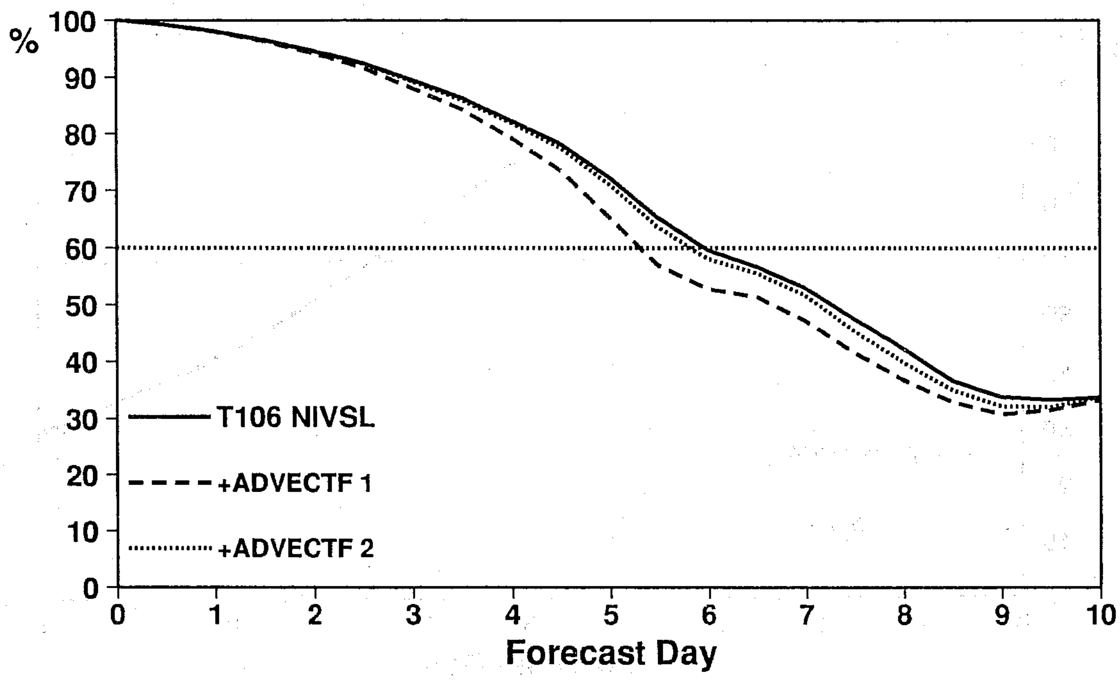


Fig 1 Anomaly correlation and r.m.s. error scores for the 500hPa height field over the Northern Hemisphere, averaged over 6 cases. Solid line, standard formulation; dashed line, with semi-implicit Coriolis terms.



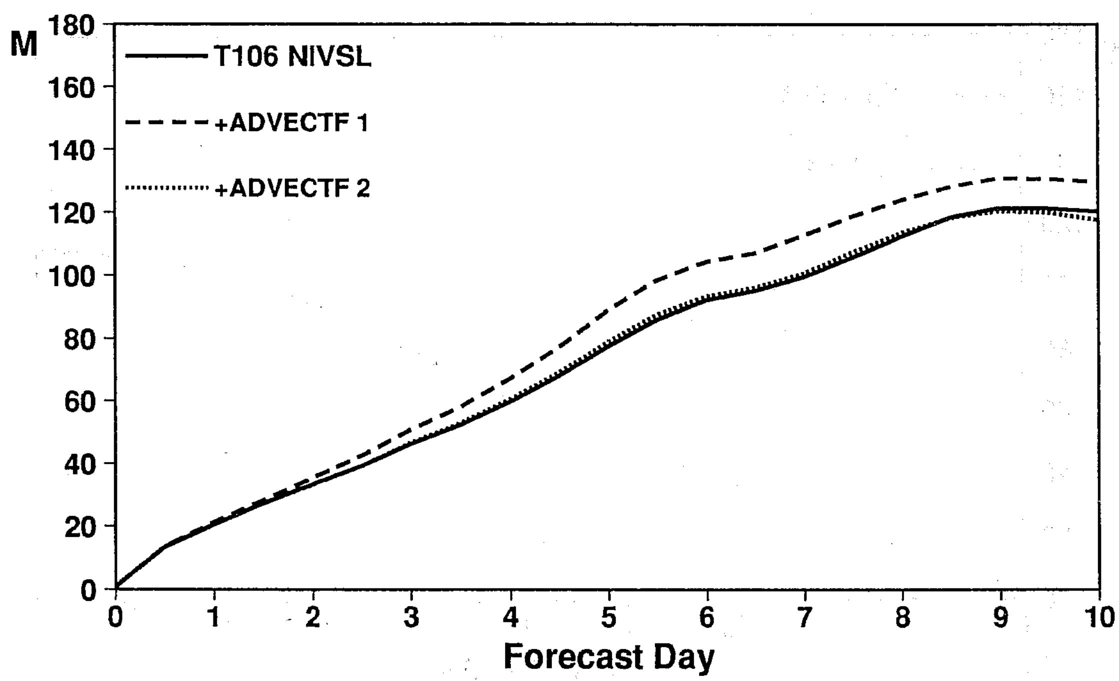


Fig 2 Anomaly correlation and r.m.s. error scores for the 500hPa height field over the Southern Hemisphere, averaged over 4 cases, using the "old" (pre-IFS) version of the model. Solid line, standard formulation; dashed line, with Coriolis terms absorbed into the advection; dotted line, the same but with more accurate spherical geometry.

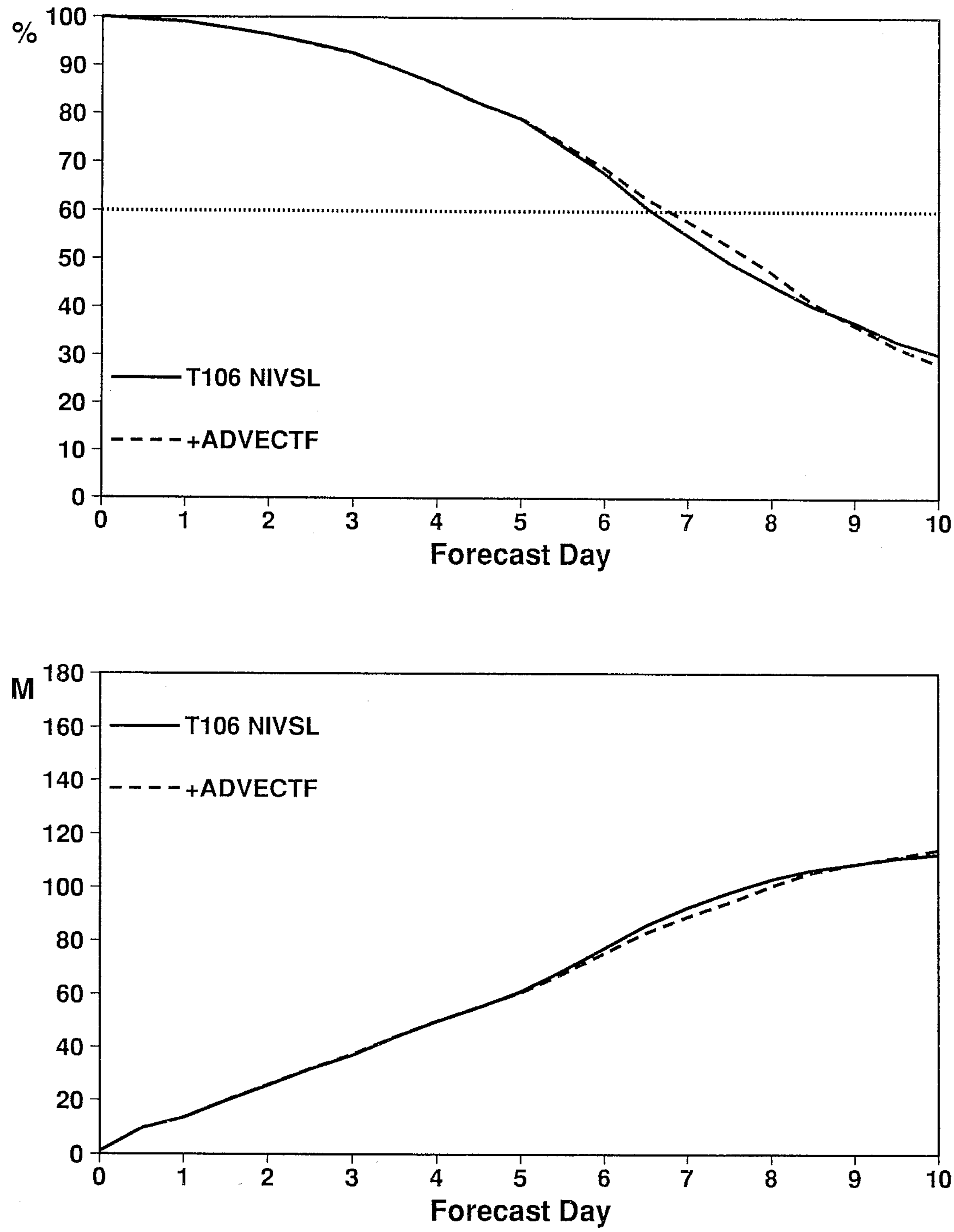


Fig 3 Anomaly correlation and r.m.s. error scores for the 500hPa height field over the Northern Hemisphere, averaged over 6 cases, using the IFS version of the model. Solid line, standard formulation; dashed line, with Coriolis terms absorbed into the advection.