TECHNICAL MEMORANDUM

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Notes on the exact solution of moist updraught equations

Peter A.E.M. Janssen and Martin Köhler

Research Department

16 July 2007

European Centre for Medium-Range Weather Forecasts
Europäisches Zentrum für mittelfristige Wettervorhersage
Centre européen pour les prévisions météorologiques à moyen terme

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European Centre for Medium-Range Weather Forecasts
Shinfield Park, Reading, RG2 9AX, England

Abstract

The entraining plume model is at the center of most convection parametrizations in global climate and weather prediction models. Entrainment can be formulated in terms of a relaxation time scale which makes it inversely dependent on the parcel vertical velocity. The result is a rather complex system of equations with a singularity for zero vertical velocity and infinite entrainment. This makes it impossible to numerically solve the coupled equations. The determination of the top of convection or the boundary layer will therefore be highly inaccurate. Here the system of equations is solved for dry, moist and cloudy situations.

1 Introduction

Cloud top height and cloud amount strongly impacts infra-red and solar radiative fluxes and therefore the Earth's energy budget. The simulation of cloud top height and cloud amount in global models is in many atmospheric conditions related to the description of convective processes. A variety of mass-flux and higher-order parametrization approaches have been proposed. However, even on the parametrization of convection there is no consensus (Arakawa, 2004). Randall et al. (2003) postulated a parametrization deadlock motivating them to circumvent the problem by attempting to resolve convection in global models at great cost. At the heart of this frustration is the entraining plume model of convective updraughts. This paper solves two common sets of plume models to allow a better understanding of these models and to simplify it's application in numerical models.

The concept of an entraining plume as a model for atmospheric convection was first proposed by Stommel (1947). Interestingly, he based his thoughts on oceanic jet theory by Rossby (1936). As still customary today he assumed that the updraught is well-mixed. The resulting horizontally discontinuous variables refer to a top-hat distribution. This idea was revived in the sixties (e.g. Squires and Turner, 1962) and applied to general circulation models (GCMs) in the seventies. Extensions are still operationally used such as the multi-plume Arakawa and Schubert (1974) convection scheme or the bulk shallow and deep convection scheme by Tiedtke (1989). More recently, mass-fluxes were shown to account for about 80% of the total fluxes in shallow convection (Siebesma and Cuijpers, 1995) and 60% in dry PBL and stratocumulus situations (Schumann and Moeng, 1991). Various approaches describe the remainder with K-diffusion or higher order terms (e.g. Siebesma et al 2007 and Lappen and Randall (2001).

Mass-flux schemes require knowledge of the updraught property ϕ_u in the mass-flux (M) term of the tendency equation for the mean environmental property $\overline{\phi}$, i.e.

$$\frac{\partial \overline{\phi}}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial z} M\left(\phi_u - \overline{\phi}\right).$$

The fractional entrainment and detrainment are then defined as the exchange of environmental and convective air masses,

$$\frac{1}{M}\frac{\partial M}{\partial z} = \varepsilon - \delta.$$

Note that entraining plume models are not only used in mass-flux approaches but are also used to determine the top of the planetary boundary layer (PBL) or convection, where the vertical velocity or buoyancy vanishes.

Several basic entrainment formulations have been proposed, for example

$$\begin{aligned}
\varepsilon &= \text{constant}, \\
\varepsilon &= \frac{1}{w\tau}, \\
\varepsilon &= \frac{\alpha}{z}, \\
\varepsilon &= \frac{2\beta}{R}.
\end{aligned}$$

Observations and models generally disagree with the first approach that assumes a constant ε but variations of it are still used because of it's simplicity and robustness. The second formulation postulates a time scale of entrainment (Siebesma, 1998; Neggers et al, 2002). If one chooses the overturning time-scale of a cumulus tower of height h ($w\tau = h$) one finds typical values of $\varepsilon \sim 10^{-3}m^{-1}$ for shallow convection and $\sim 10^{-4}m^{-1}$ for deep convection, which are close to observations. The third formulation $\varepsilon = \alpha/z$ postulates that the entrainment at height z is dominated by eddies of size z. This formulation works well near the surface and can also be adjusted for the top of the PBL (Siebesma et al 2007). The last entrainment formulation mentioned $\varepsilon = 2\beta/R$ with $\beta \sim 0.1$ is based on similarity arguments for plumes (Simpson et al 1965, Turner, 1973). It's rational though has been questioned (Siebesma and Cuijpers, 1995). A review of entrainment formulations is given in Siebesma (1998).

A few theoretical and numerical problems were pointed out relating to the entraining plume model in particular to the treatment at the parcel top. While investigating single column models with entrainment formulation $\varepsilon = const.$, Warner (1970) found that entraining plume models cannot simultaneously predict values of liquid water content and cloud depth in agreement with observations. Lock (2001) and Grenier and Bretherton (2001) identified a common problem in the determination of PBL top, which is crucial to determine the height of mixing and the resulting PBL top entrainment. They developed a technique called profile reconstruction in which the PBL is assumed well-mixed and the free troposphere is assumed to have a fixed lapse rate. The PBL top can then be reconstructed between model layers. Moeng and Randall (1984) documented a spurious oscillation in higher-order single column models. A large artificial damping had to be applied to solve this problem. At ECMWF we decided to adopt the $\varepsilon = 1/w\tau$ formulation as part of our Eddy-Diffusivity Mass-Flux framework implementation for PBL parametrization. But we discovered a long-standing problem of over-prediction of PBL height and related overestimates of PBL top entrainment. We found that this problem was related to insufficient parcel entrainment near PBL top, where the parcel should slow down to $w = 0$. This results in a singularity of infinite entrainment, which cannot be properly handled numerically.

We found a solution to this problem by obtaining the exact solution of the coupled plume equations (cf. e.g. Siebesma and Holtslag, 1996) for $\varepsilon = 1/w\tau$. Those equations describe the fate of the properties of an 'updraught' fluid element when it moves around in a background atmosphere with a given liquid water potential temperature $\bar{\theta}_l(z)$, virtual potential temperature $\bar{\theta}_v(z)$ and total water mixing ratio $\bar{q}_t$. This paper derives this solution and discusses it's implications.

Before we write down the relevant equations first a remark on notation. The background quantities will be denoted by a bar over the relevant character (i.e. $\bar{\theta}$), while the quantities for the fluid element do not have an over-bar. Liquid and water vapour are denoted with subscripts l and v.

The equation for the updraught velocity w follows from the momentum equation:

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\left(\frac{1}{\tau} + \varepsilon|w|\right)w + g\frac{\theta_v - \bar{\theta}_v(z)}{\bar{\theta}_v(z)}, \tag{1}$$

which describes effects of entrainment and buoyancy on the vertical movement of a parcel. Here, the effects of entrainment are represented by two terms. The first term has been proposed by Siebesma (1998) and Neggers et al (2002) and it is given by a relaxation term with constant relaxation time τ. The second term, $\varepsilon|w|w$, is basically a turbulent drag with constant drag coefficient/entrainment rate $\varepsilon =$ constant. As is common in parametrizations of turbulence, the drag involves the product of the absolute value of the velocity and the velocity vector.

In the absence of entrainment there are two invariants for moist air, namely the total water mixing ratio $q_t = q_v + q_l$ and the liquid water static energy s_l of the parcel, defined by

$$s_l = c_p T + gz - L_v q_l,$$

where c_p is the specific heat at constant pressure, g is acceleration of gravity, and L_v is the latent heat of vaporisation. Alternatively, one may introduce the potential temperature θ according to

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa},$$

with $\kappa = R/c_p$, R the gas constant, p the pressure at height z and p_0 the reference pressure (1000hPa). For hydrostatic balance, $\mathrm{d}p/\mathrm{dz} = -\rho g$, and using the gas law $p = \rho RT$ it can be shown that invariance of liquid water potential temperature θ_l follows from invariance of s_l with

$$\theta_l = \theta - \frac{L_v \theta}{c_p T} q_l.$$

We use θ_l for convenience in the following.

In the presence of entrainment, the equation for θ_l becomes

$$\frac{\partial \theta_l}{\partial t} + w\frac{\partial \theta_l}{\partial z} = -\left(\frac{1}{\tau} + \varepsilon|w|\right)\left(\theta_l - \bar{\theta}_l(z)\right) \tag{2}$$

while the equation for the total water mixing ratio q_t becomes

$$\frac{\partial q_t}{\partial t} + w\frac{\partial q_t}{\partial z} = -\left(\frac{1}{\tau} + \varepsilon|w|\right)\left(q_t - \bar{q}_t(z)\right) \tag{3}$$

In this note we discuss exact, steady-state solutions of the moist updraught equations (1), (2) and (3). The case of finite turbulent drag ε and infinite relaxation time τ is straightforward to solve and will not be discussed in detail here (As a reference the relevant solution is given in Appendix C). On the other hand, the case of finite relaxation time and vanishing turbulent drag ($\varepsilon = 0$) is by no means trivial because one deals with a nonlinearly coupled system. Furthermore, this case is of particular interest because it has a singularity for vanishing vertical velocity. Therefore, in this note the relaxation model for entrainment will be discussed extensively, and it will be seen that the singularity for vanishing vertical velocity imposes a robust structure on the solution. The Sections 2, 3 and 4 will discuss the exact solution for a dry atmosphere, a moist, non-cloudy atmosphere and a cloudy atmosphere with constant background. In Section 5 we show how to deal with the case of stratification in the atmosphere. In particular, we discuss the behaviour of a parcel near an inversion for increasing inversion strength. For reasonable strong inversions the solution for vertical velocity will show a cat's-eye pattern, suggesting that, because a parcel may execute a 'damped' oscillation around the inversion for a long time, there could be a considerable amount of heat and momentum exchange between the parcel and the background, which may lead to erosion of the inversion. In the final section we present for the case of a dry atmosphere an exact solution which combines the effects of relaxation and turbulent diffusion.

The buoyancy term in our main equation (1) is in some sense the difficult one because it depends on the virtual potential temperature θ_v of the updraught and the background virtual potential temperature $\overline{\theta_v}(z)$ (which is assumed to be given), while the relevant temperature equation is in terms of the liquid water potential temperature θ_l. The virtual potential temperature θ_v can be written as a function of conserved variables liquid water potential temperature θ_l and total water mixing ration q_t (see App. B). This relationship depends on the circumstances, and therefore we will distinguish between the cases of a dry atmosphere, a moist, non-cloudy atmosphere and a cloudy atmosphere.

2 Exact solution for dry atmosphere

For a **dry** atmosphere, q_v and q_l vanish, hence the potential and virtual potential temperature are identical. The relevant updraught equations become:

$$w\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{w}{\tau} + g\frac{\theta - \overline{\theta}(z)}{\theta(z)}, \tag{4}$$

$$w\frac{\mathrm{d}\theta}{\mathrm{d}z} = -\frac{1}{\tau}\left(\theta - \overline{\theta}(z)\right) \tag{5}$$

In order to solve the above problem we assume that the background profiles are independent of height. Introduce the dimensionless quantities T, W, and Z according to

$$T = \frac{\theta - \overline{\theta}}{\overline{\theta}}, W = \frac{w}{g\tau}, Z = \frac{z}{g\tau^2},$$

then Eqns. (4-5) become

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -1 + \frac{T}{W}, \tag{6}$$

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -\frac{T}{W}. \tag{7}$$

This is a second-order nonlinear problem and, at first instance, one might not be very hopeful in finding an exact solution. Fortunately, however, this problem has an adiabatic invariant. This is readily seen by adding (6) and (7) with the result

$$\frac{\mathrm{d}\left(T+W\right)}{\mathrm{d}Z} = -1,$$

hence we find the invariant

$$T + W + Z = \alpha, \tag{8}$$

where α is an integration constant to be determined by the boundary conditions. Elimination of W from (7) then gives a first-order, nonlinear differential equation for T,

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = \frac{T}{T + Z - \alpha}, \tag{9}$$

This first-order problem may be integrated immediately (cf. Eq. (A21) of the Appendix), and the general solution for X becomes the simple relation

$$T \log \beta T = Z - \alpha, \tag{10}$$

where β is another integration constant.

The integration constants α and β are now determined by means of the boundary condition on T and W at the surface $Z = 0$,

$$T(0) = T_0,\ W(0) = W_0.$$

Using Eq. (8) one finds for α,

$$\alpha = T_0 + W_0,$$

while from the solution (10) one finds

$$\beta = \frac{1}{T_0}\, e^{-\alpha/T_0}.$$

The top of the boundary layer is now defined as that height Z_T where the vertical velocity W vanishes. Hence, from the condition $W(Z = Z_T) = 0$ one finds from (8) that

$$T_T = \alpha - Z_T, \tag{11}$$

and using this in the exact solution (10) one finds for Z_T

$$(Z_T - \alpha)\,[1 + \log\beta(\alpha - Z_T)] = 0,$$

which gives two conditions on Z_T, namely

$$Z_T = \alpha,\ \text{or}\ ,\ Z_T = \alpha - \frac{1}{\beta e}. \tag{12}$$

Substituting the first condition, $Z_T = \alpha$, in Eq. (11) immediately gives the vanishing of the buoyancy, whereas according to the second condition of (12) the buoyancy remains finite. In other words, at $Z_T = \alpha$ both buoyancy and vertical velocity of the parcel vanish, and therefore one would expect a stable condition, while when $Z_T = \alpha - 1/\beta e$ the parcel still experiences a finite buoyancy $T_T = 1/\beta e$, hence there is potential for instability, i.e. the second condition is not an equilibrium point. Because of this stability consideration one would expect that the top of the boundary layer or convection is given by $Z_T = \alpha$.

Now, using the expressions for α and β one finds that the first condition simply becomes,

$$Z_T = T_0 + W_0.$$

In terms of dimensional variables one therefore finds that the height of the boundary layer is entirely determined by the surface conditions in the following manner:

$$z_T = g\tau^2\frac{\theta_0 - \bar{\theta}}{\theta} + w_0\tau, \tag{13}$$

an extremely elegant and useful result indeed. Useful, because in practice the height of the boundary layer is an important quantity to know (In typical cases, by the way, the boundary layer height is mainly determined by the buoyancy term).

It is remarked, however, that the result (13) only holds for positive boundary conditions. In general, one needs to make a distinction between cases with different signs of T_0 and W_0 (or $\theta_0 - \bar{\theta}$ and w_0). These cases will be discussed during the graphical construction of the solution.

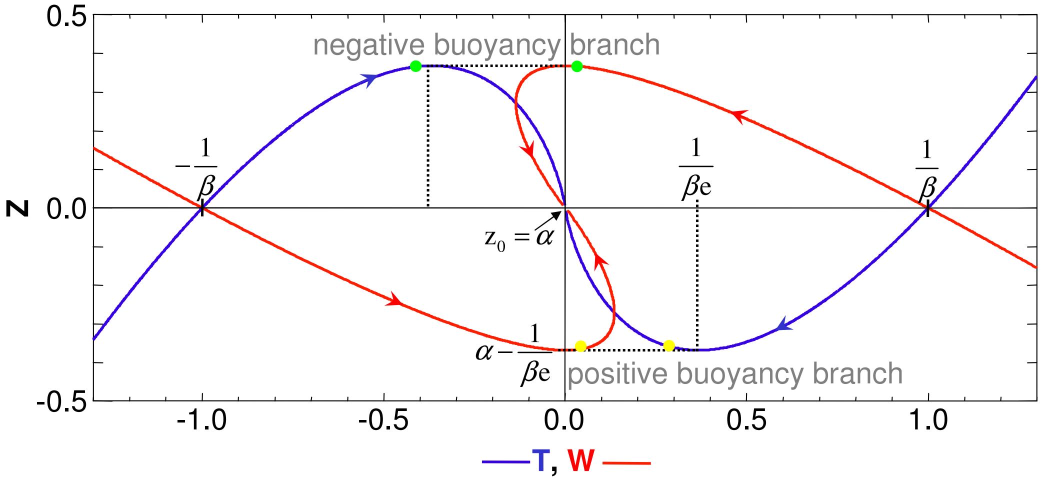

Figure 1: All dry parcel solutions. See text for explanation.

2.1 Graphical construction of the solution

Although T is not given explicitely in terms of the height Z, the relation $T(Z)$ is obtained from Eq. (10) by determining Z for given T in a reasonable range for T.

As indicated in the previous section, there are several cases to be distinguished, depending on the sign of the boundary conditions. All cases considered can be represented in one graph. Fig. 1 shows two solutions, one branch corresponding to positive buoyancy, called T_+, and one corresponding to negative buoyancy, T_-. The origin, where $z = \alpha$ and $T = W = 0$, represents the singularity where all parcels come to a permanent halt ($T = 0$ means zero acceleration). This point therefore splits the two solution branches.

First, we follow parcels representing the T_+ branch. Note that those parcels are constantly accelerated upwards while consuming and reducing their buoyancy. Depending on the initial conditions they can start at large negative (downward) velocities W and large positive buoyancy T. They reach $Z = \alpha$ the first time at $T = -W = 1/\beta$. Later they pass through their lowest point at $Z = \alpha - 1/\beta e = -0.368$ where $W = 0$ and $T = 1/\beta e$. The remaining buoyancy accelerates the parcels into an upward motion. Their maximum upward velocity is reached at $Z = \alpha - 2/\beta e^2$ and $T = W$. Then entrainment destroys both T and W until the parcels come to a rest at $Z = \alpha$.

The negative buoyancy branch follows a path that is symmetric with respect to the positive buoyancy branch with large upward motion first, reaching a maximum height and then moving downwards to come to rest at $Z = \alpha$.

Interestingly, in this graph Z, T and W are scaled with $1/\beta$ and Z is shifted by α. This means that it preserves it's shape and a table can be easily pre-calculated for the full range of values.

We now look once more at two typical scenarios of PBL parcels. First take a surface layer perturbed parcel with excess temperature of 1K and an upward velocity of 1 m/s ($T = (\theta - \overline{\theta})/\overline{\theta} \approx 1/300$, $W = w/g\tau \approx 1/4000$). This initial condition is indicated by the yellow points on the T_+ branch. Buoyancy dominates that initial state by a factor of 10. It is then converted into upward velocity until entrainment reduces both T and W towards zero.

Second, take an upward moving parcel (1 m/s) that reaches into the stable entrainment zone on top of the PBL with negative buoyancy ($\Delta T = -10K; T = (\theta - \overline{\theta})/\overline{\theta} \approx -1/30$, $W = w/g\tau \approx 1/4000$). Its initial conditions are indicated by the green dots on the $T\_$ branch in Fig. 1. It is quickly accelerated downward and reaches a resting point below injection. This parcel trajectory is called overshoot. The last part of it represents a downdraught.

3 Exact solution for non-cloudy, moist atmosphere

For the **non-cloudy** case there is only water vapour and no liquid. In that event, according to Eq. (4) the virtual potential temperature only depends on the vapour mixing ratio,

$$\theta_v = \theta\,(1+0.61q_v)$$

and the relevant equations become:

$$\begin{aligned}
w\frac{\mathrm{d}w}{\mathrm{d}z} &= -\frac{w}{\tau}+g\frac{\theta(1+0.61q_v)-\bar{\theta}_v(z)}{\theta_v(z)}, \\
w\frac{\mathrm{d}\theta}{\mathrm{d}z} &= -\frac{1}{\tau}\left(\theta-\bar{\theta}(z)\right), \qquad (14)\\
w\frac{\mathrm{d}q_v}{\mathrm{d}z} &= -\frac{1}{\tau}\left(q_v-\bar{q}(z)\right)
\end{aligned}$$

Now we are dealing with a third order nonlinear problem, but even this case can be solved exactly. To proceed we introduce dimensionless quantities

$$T=\frac{\theta-\bar{\theta}}{\theta},\; Q=\frac{q_v-\bar{q}}{\bar{q}},\; W=\frac{w}{g\tau},\; Z=\frac{z}{g\tau^2}$$

and for uniform background profiles the starting set of equations assumes the form

$$\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}Z} &= -1+\frac{1}{W}\left[T+aQ\,(T+1)\right], \\
\frac{\mathrm{d}T}{\mathrm{d}Z} &= -\frac{T}{W}, \qquad (15)\\
\frac{\mathrm{d}Q}{\mathrm{d}Z} &= -\frac{Q}{W}, \qquad (16)
\end{aligned}$$

where

$$a=\frac{0.61\,\bar{q}}{1+0.61\,\bar{q}}$$

is given.

As a first step towards the solution consider Eqns. (15) and (16) together and eliminate W between them. Then, we find

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}Z} = \frac{1}{Q}\frac{\mathrm{d}Q}{\mathrm{d}Z}\rightarrow\frac{Q}{T}=\text{ const }=\gamma,$$

hence we may eliminate Q in the equation for the vertical motion with the result that we are left with the second-order system

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -1+\frac{1}{W}\left[A\,T+B\,T^2\right], \qquad (17)$$

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -\frac{T}{W}, \qquad (18)$$

where $A = 1 + a\gamma$ and $B = a\gamma$. Comparing this with the dry atmosphere case, Eqns. (6)- (7), it is seen that the only essential difference is a T^2-term in the buoyancy term.

Once more, we can find an 'invariant'. To see this we eliminate the $1/W$-term in (17) by using Eq. (18), hence

$$\frac{dW}{dZ} = -1 - \frac{dT}{dZ}(A + BT),$$

and integrating once one finds the 'invariant'

$$W = -Z - AT - \frac{B}{2}T^2 + \alpha, \tag{19}$$

where α is an integration constant. This is the equivalent of Eq. (8), but now there is a quadratic relation between vertical velocity and dimensionless temperature.

From Eq. (18) we now obtain a first-order equation for the temperature by eliminating W with (19) and the result is

$$\frac{dT}{dZ} = \frac{T}{AT + BT^2/2 + Z - \alpha},$$

which is the equivalent of Eq. (9).

The general solution of this problem is obtained in the Appendix (cf. Eq. (A22). It reads

$$(1 + a\gamma)\,T\log(\beta T) + \frac{a\gamma}{2}T^2 = Z - \alpha, \tag{20}$$

which, because it is a third-order problem, involves three integration constants α, β and γ.

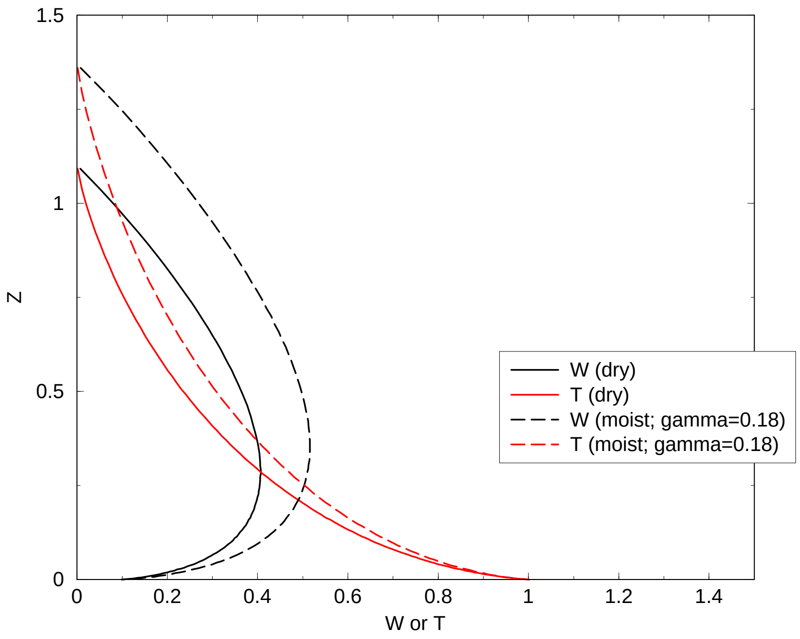

Figure 2: Effect of moisture on the T_+ type solution for updraught (full lines, dry; dashed line, moist) for $a\gamma = 0.18$ The temperature difference at the surface $T_0 = 1$ while the vertical velocity $W_0 = 0.1$.

The integration constants α and β are now determined by means of the boundary condition on T and W at the surface $Z = 0$,

$$T(0) = T_0,\; W(0) = W_0.$$

Using Eq. (19) one finds for α,

$$\alpha = W_0 + AT_0 + \frac{B}{2}T_0^2, \tag{21}$$

while from the solution (20) one finds

$$\beta = \frac{1}{T_0}\exp(-\frac{\alpha + BT_0^2/2)}{AT_0}).$$

Finally, defining the top of the boundary layer as that height Z_T where the vertical velocity W and temperature perturbation vanish, one finds immediately from (19) that

$$Z_T = \alpha. \tag{22}$$

In Fig. 2 the impact of moisture on the T_+ type solution is shown for the 'typical' case of $a\gamma = 0.18$. [1] The effects of moisture on the behaviour of a parcel are relatively modest. Moisture tends to increase buoyancy and hence gives rise to an increase of vertical velocity as is evident from the increased maximum in Fig. 2. Furthermore the boundary layer height increases as is also evident from the expressions of α (21) and Z_T (22).

4 Exact solution for cloudy atmosphere

Finally, the **cloudy** case is the most complicated one. The mixing ratio q_t is now the sum of liquid water and water vapour mixing ratio,

$$q_t = q_v + q_l$$

but since we are in a cloud the vapour mixing ratio is given by the saturation value,

$$q_v = q_{sat} \sim \frac{R_{dry}}{R_{vap}}\frac{A}{p}e^{-B/T}. \tag{23}$$

This approximation follows from the Clausius-Clapeyron equation when assuming the latent heat as constant and is accurate to about 1% within ±25°C (Rogers and Yau, 1989, $A_{liq} = 2.54\cdot10^{11}$Pa, $B_{liq} = 5.42\cdot10^3$K, $A_{ice} = 3.41\cdot10^{12}$Pa, $B_{ice} = 6.13\cdot10^3$K). Therefore, the liquid water mixing ratio can be expressed in terms of the total water mixing ratio q_t temperature and pressure

$$q_l = q_t - q_{sat}(p,T).$$

Yet temperature in return depends on condensation and therefore q_l. An iterative solution is required to express θ, q_l and θ_v in terms of θ_l and q_t: $q_l = q_l(\theta_l, q_t)$, which are required in the cloudy buoyancy term

$$B = g\frac{\theta_v - \bar{\theta}_v(z)}{\theta_v(z)} = g\frac{\theta\left(1 + 0.61q_t - 1.61q_l\right) - \bar{\theta}_v(z)}{\theta_v(z)}$$

The relevant equations become for the cloudy case:

$$w\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{w}{\tau} + B(\theta_l, q_t), \tag{24}$$

$$w\frac{\mathrm{d}\theta_l}{\mathrm{d}z} = -\frac{1}{\tau}\left(\theta_l - \bar{\theta}_l(z)\right) \tag{25}$$

$$w\frac{\mathrm{d}q_t}{\mathrm{d}z} = -\frac{1}{\tau}\left(q_t - \bar{q}_t(z)\right) \tag{26}$$

[1] We find $a \simeq 0.006$ for $\langle q_v \rangle = 0.01$, while $\gamma = 30$ for a perturbation Q of 10% and a relative perturbation temperature of $1/300$.

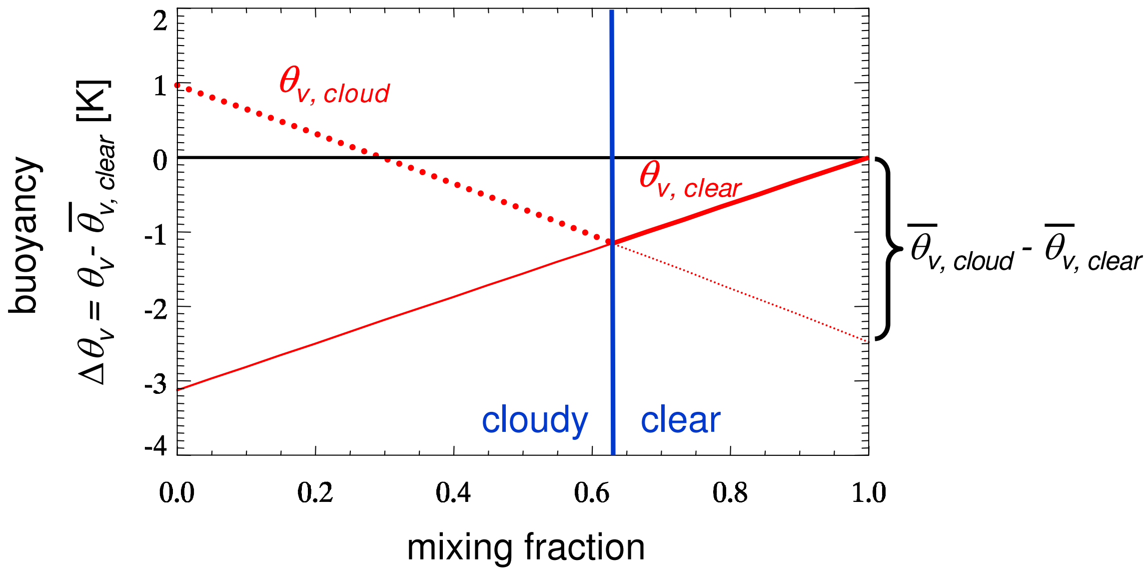

Figure 3: Virtual potential temperature difference between an entraining parcel and the environment as a function of mixing fraction between undiluted air and the environment (see also Kain and Fritsch, 1990). Both virtual potential temperature formulations for cloudy ($\theta_{v,cloud}$, dotted red line) and clear ($\theta_{v,clear}$, solid red line) conditions are drawn. The saturation threshold (blue line) indicates which virtual potential temperature formulation is applicable. On the right side (environmental conditions) the first term in Eq. (B22) is diagnosed. The values for this typical example were chosen as p=900 hPa, θ_l=286 K, $\overline{\theta}_l$=290 K, q_t=10 g/kg, $\overline{q}$=5 g/kg.

We proceed in a similar manner as in the previous section and introduce dimensionless quantities

$$T = \frac{\theta_l - \overline{\theta}_l}{\theta_l},\; Q = \frac{q_t - \overline{q}}{\overline{q}},\; W = \frac{w}{g\tau},\; Z = \frac{z}{g\tau^2}. \tag{27}$$

For uniform background profiles the set of equations becomes

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}Z} &= -1 + \frac{1}{W}B(T,Q), \\ \frac{\mathrm{d}T}{\mathrm{d}Z} &= -\frac{T}{W}, \\ \frac{\mathrm{d}Q}{\mathrm{d}Z} &= -\frac{Q}{W}. \end{aligned}$$

Noting that by means of the last two equations it can be shown that Q and W are related to each other according to

$$Q = \gamma T, \tag{28}$$

our set of equations reduces to

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}Z} &= -1 + \frac{1}{W}B(T,\gamma T), \\ \frac{\mathrm{d}T}{\mathrm{d}Z} &= -\frac{T}{W}, \end{aligned}$$

which has the same form as the problem we have solved exactly in the Appendix A.

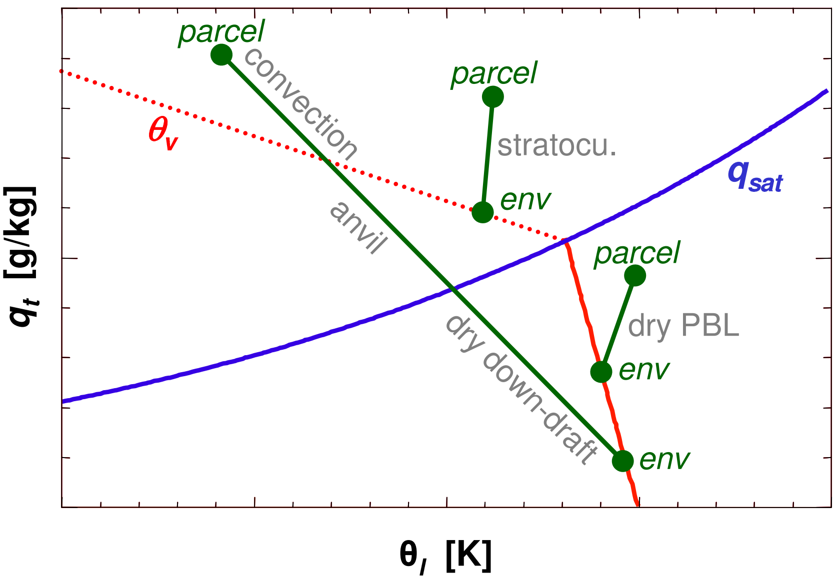

Figure 4: $\theta_v = const$ (red) as a function of conserved variables for clear (solid line) and cloudy (dotted line) conditions. $q_t = q_{sat}$ is denoted by the blue line where all cloudy points are above that line.

In Appendix B we have studied in detail the particular form of the buoyancy term for the stratocumulus case and the cumulus case. The stratocumulus case is here defined as convection within a cloudy environment, while the cumulus cloud is assumed to rise within a clear environment. In general we find the following form

$$B(T,Q) = a + bT + cQ + dQT + eQ^2, \tag{29}$$

which upon using Eq. (28) becomes

$$B(T) = C + AT + BT^2,$$

where $C = a$, $A = b + c\gamma$ and $B = d\gamma + c\gamma^2$. It is interesting to note that the only difference between the stratocumulus and the cumulus case is the constant term C, which is the difference between the cloudy and clear formulations of θ_v given the environmental conditions $(\overline{q}, \overline{\theta}_l)$. This term can be illustrated by drawing the virtual potential temperature as a function of mixing fraction (see Fig. 3). This mixing fraction corresponds to the convective mixing line in Fig. 4 (green line "convection"). Note that this term is always negative and remains constant throughout the entrainment process because it only depends on the environment.

The constant term gives rise to a new structure of the solution because there is still finite, negative buoyancy for vanishing potential temperature. In other words, the point $W = 0$, $T = 0$ is not an equilibrium point anymore. Rather, after the parcel reaches maximum height for zero velocity it will return to the earth's surface because of the negative buoyancy for small temperature T. The solution in this case is obtained in Appendix A and reads

$$Z - \alpha = -C\,(1 + \log T) + AT \log(\beta T) + \frac{B}{2}T^2. \tag{30}$$

It is depicted in Fig. 5, which for a relatively small value of $C = -0.1$ shows already dramatic differences with the dry case. In the dry case there is an equilibrium point corresponding with the top of the boundary layer, while in the cloudy, cumulus case the height of the boundary layer is considerably smaller and does not correspond to an equilibrium point (as for vanishing velocity there is still negative buoyancy). As a consequence the parcel returns to the earth's surface.

The solution for the stratocumulus case (hence no constant term) is qualitatively similar to the non-cloudy, moist atmosphere example of the previous section. The essence of the solution is therefore depicted in Fig. (2).

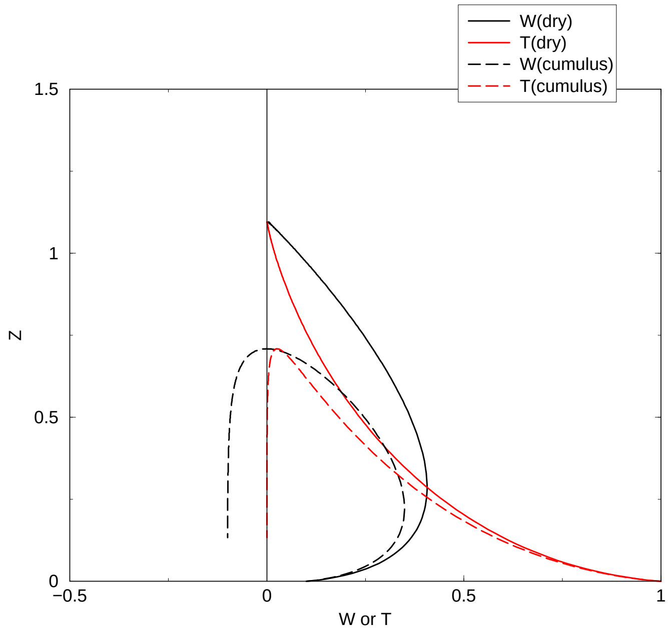

Figure 5: Comparison of solution in case of cumulus convection with the T_+ type solution for updraught (full lines, dry; dashed line, cumulus) for $C = 0.1$ The temperature difference at the surface $T_0 = 1$ while the vertical velocity $W_0 = 0.1$.

5 Effect of varying background profiles

It is difficult to obtain the exact solution for T and W for general background profiles, except when it is assumed, as usually done in numerical models, that in a layer of thickness Δz the potential temperature $\overline{\theta}$ is constant while jumps are allowed from one layer to the next. Jump profiles can still be dealt with in the context of the full nonlinear problem as inside the i^{th} layer with mean height z_i the general solution is still given by Eq. (30) but now with integration constants α_i, β_i and γ_i, hence

$$Z - \alpha_i = -C\,(1 + \log T) + A(\gamma_i)T\log(\beta_i T) + \frac{B(\gamma_i)}{2}T^2,\ Z \in \mathscr{D}, \tag{31}$$

where T is the dimensionless temperature in the i^{th} layer with mean dimensionless height Z_i, and where the domain $\mathscr{D} = [Z_i - \Delta Z/2, Z_i + \Delta Z/2]$. The main problem then to solve is how to connect the integration constants from one layer to the next. These connection formulae will be derived here from the equations for vertical motion, potential temperature and mixing ratio.

Our starting point is the set of Eqns. (24)-(26), and we introduce the dimensionless variables T, Q and W according to (27), but now we allow the background profile to be a function of height z. In stead of the set (24)-(26) one now finds

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -1 + \frac{1}{W}B(T,Q), \tag{32}$$

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -\frac{T}{W} - (T+1)\,\frac{\mathrm{d}}{\mathrm{d}Z}\log\overline{\theta}(Z), \tag{33}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}Z} = -\frac{Q}{W} - (Q+1)\,\frac{\mathrm{d}}{\mathrm{d}Z}\log\overline{q}(Z), \tag{34}$$

We solve this nonlinear set of equations for jump profiles in $\overline{\theta}$ and $\overline{q}$. Specifically, we write for $\overline{\theta}$ and $\overline{q}$ near

$Z_{i+1/2} = Z_i + \Delta Z/2$

$$\overline{\theta} = \overline{\theta}_i + \Delta\theta_i\,\mathrm{H}(Z - Z_{i+1/2}), \quad \overline{q} = \overline{q}_i + \Delta q_i\,\mathrm{H}(Z - Z_{i+1/2}),$$

where H denotes the Heavyside function, $\Delta\theta_i = \overline{\theta}_{i+1} - \overline{\theta}_i$, and $\Delta q_i = \overline{q}_{i+1} - \overline{q}_i$. Clearly, the jumps in the background profiles give rise to δ-function singularities in the equations for T and Q and therefore only 'weak', discontinuous solutions can be obtained, i.e. T and Q have jumps as well. However, as the equation for W does not contain a singularity, the vertical velocity is found to be continuous (but the first derivative shows, of course, jumps)

The connection formulae for α_i, β_i and γ_i now follow from the jump conditions for W, T and Q. For example, the jump condition for T follows from an integration of Eq. (33) over the jump in $\overline{\theta}$ located at $Z_{i+1/2} = Z_i + \Delta Z/2$. Assume that the vertical motion W is continuous across the jump. Then, integration of (33) over

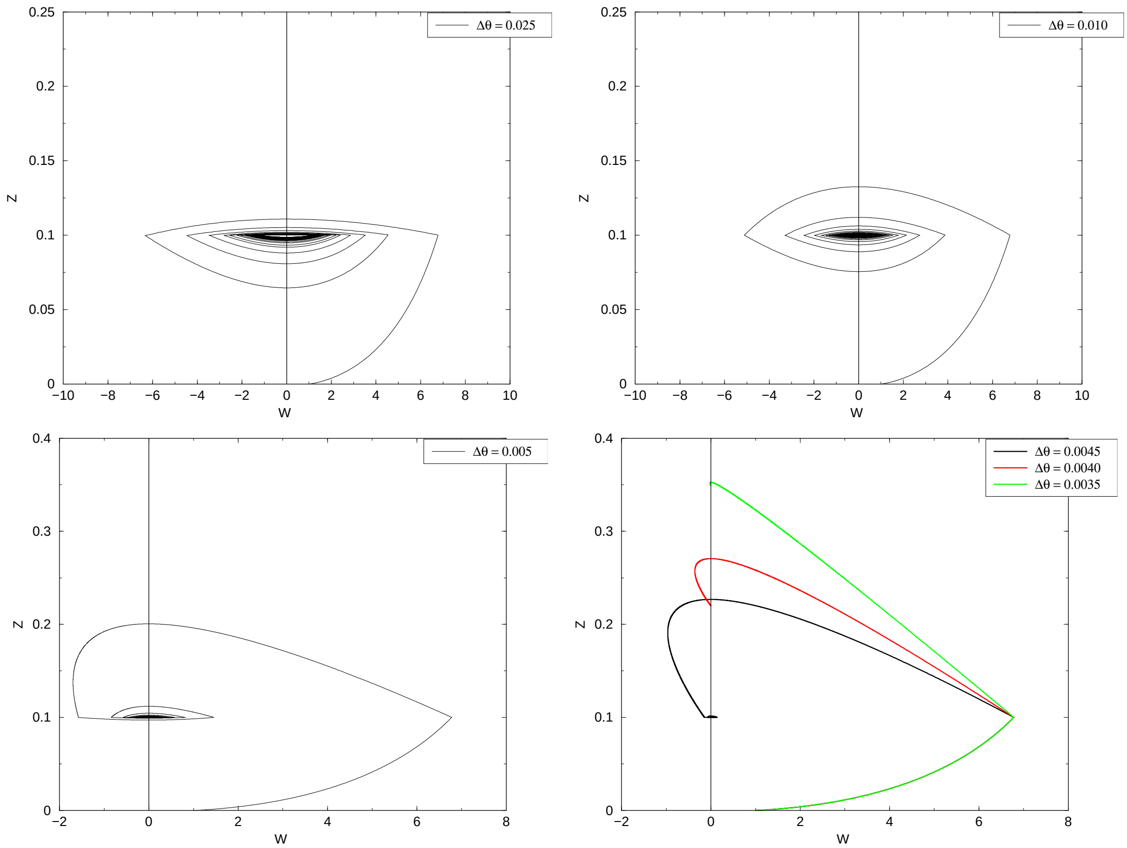

Figure 6: Exact solution for the vertical velocity in case the atmosphere has an inversion at dimensionless height $Z \simeq 0.1$. The different panels show cases of decreasing potential temperature, ranging from a strong inversion ($\Delta\theta = 0.025$) to a weak inversion ($\Delta\theta = 0.0035$).

the jump at $Z_{i+1/2}$ from $Z_{i+1/2} - \varepsilon$ to $Z_{i+1/2} + \varepsilon$, while taking the limit of small ε gives for the increment[2] $\Delta T = T(+\varepsilon) - T(-\varepsilon)$ in T

$$\Delta T = -\left(1 + \langle T \rangle\right) \frac{\Delta\theta_i}{\langle\theta\rangle}. \tag{35}$$

[2]For ease of notation the argument $Z_{i+1/2} - \varepsilon$ has been replaced by $-\varepsilon$, etc.

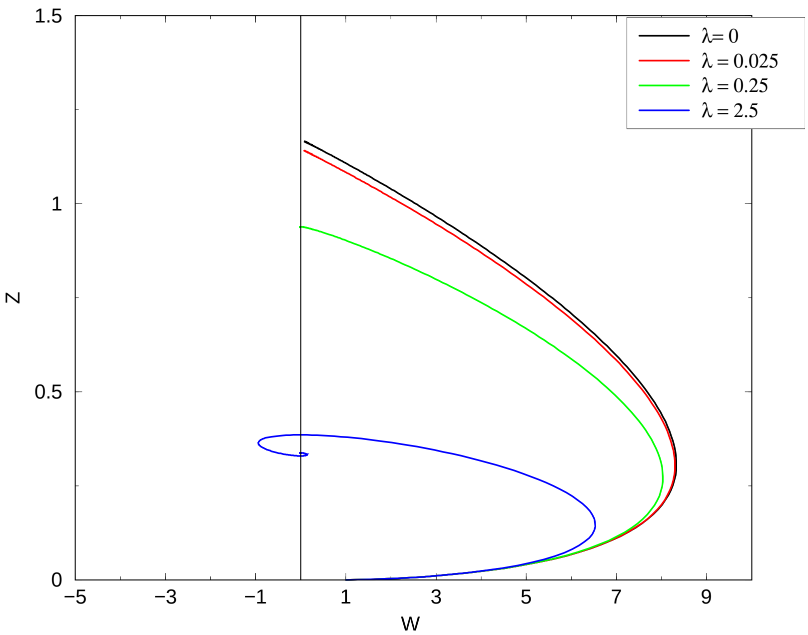

Figure 7: Exact solution for the vertical velocity in case the atmosphere has an exponential potential temperature profile. Shown are cases of increasing inverse decay rates, ranging from a realistic rate ($\lambda_\theta = 0.025$) to a a far too large value ($\lambda_\theta = 0.025$).For comparison, the case of a constant background temperature is shown as well.

Here, $\langle T \rangle = (T(-\varepsilon) + T(+\varepsilon))/2 = T(-\varepsilon) + \Delta T/2$ which still depends on ΔT. Rearranging Eq. (35) one finds explicitely for ΔT

$$\Delta T = -\frac{\Delta\theta_i}{1+\Delta\theta_i/2}\left(1+T(-\varepsilon)\right),$$

where $\Delta\theta_i$ is short-hand for $\Delta\theta_i/\langle\overline{\theta}\rangle$. Similarly, the jump in Q is found to be

$$\Delta Q = -\frac{\Delta q_i}{1+\Delta q_i/2}\left(1+Q(-\varepsilon)\right).$$

Since now the jumps in T and Q are known, while W is continuous, the integration constants in the i^{th}-layer follow from the values of W, T and Q at the bottom of the layer. For γ one finds

$$\gamma_i = \frac{Q(+\varepsilon)}{T(+\varepsilon)}, \tag{36}$$

while α_i follows most easily from the invariant (A4),

$$\alpha_i = W(+\varepsilon) + Z_{i+1/2} + AT(+\varepsilon) + \frac{1}{2}BT(+\varepsilon)^2 + C\log(|T(+\varepsilon)|). \tag{37}$$

Finally, β_i is obtained from the solution (31) with the result

$$\beta_i = \frac{1}{T(+\varepsilon)}\exp\left\{\left[C(1+\log(|T(+\varepsilon)|) + +Z_{i+1/2} - \alpha_i - 1/2BT(+\varepsilon)^2)\right]/(AT(+\varepsilon))\right\} \tag{38}$$

By making use of (36)–(38) one may continue the solution from one layer to the next. As an example, we show in Fig. 6 the case of a two-layer fluid where the increment in background potential temperature is chosen to decrease from $\Delta\theta_i/ = 0.025$ to 0.0035 at a dimensionless inversion height $Z = 0.1$. For a small increase in background temperature in the second layer, buoyancy is only slightly reduced giving a small modification to the solution for W of the parcel. However, increasing the background temperature increment in such a way that the parcel gets sufficient negative buoyancy in the second layer, there are dramatic consequences for the solutions: in the second layer we now get a T_{-} type solution while the equilibrium point starts to move towards the inversion height . Increasing the temperature increment even more the parcel returns to the first layer and starts executing a damped oscillation. However, the details of this damped cat's-eye pattern depend to a large extend on the resolution of the calculations. In these particular examples we have chosen a temperature increment $\Delta T = 0.001$, while in all other calculations in this paper we took $\Delta T = 0.03$.

While the parcel is executing its oscillations around the inversion height there is ample time for exchange of heat and momentum between the parcel and the background atmosphere and one would expect that because of this interaction the inversion will be eroded. Clearly, this parcel-background interaction has not been taken into account yet and it will be of interest to see what will happen to the fate of the parcel, for example whether the oscillation gets more damped or not.

Finally, it is of interest to study the impact of a slowly varying background profile on the solution for the vertical velocity of the parcel. To that end we took an exponential potential temperature and mixing ratio profile,

$$\bar{\theta} = e^{\lambda_\theta Z}, \ \bar{q} = e^{\lambda_q Z},$$

and results for $\lambda_q = 0$ and increasing λ_θ from a realistic value of 0.025 to a highly unrealistic value of 2.5 are shown in Fig. 7. Impact of stratification on the velocity profile is for realistic inverse decay rates marginal. Only for very strong stratification the parcel will execute an heavily damped oscillation, similarly to the case of a strong inversion.

6 Final comments

In this note we have studied properties of the moist updraught equations with an entrainment parametrization that follows from a simple relaxation model with a constant relaxation time τ. The essential feature of the exact solutions of this model for updraught is that, apart from the 'cumulus' case, there exist a 'true' equilibrium point in the solution with vanishing buoyancy and vertical velocity. The reason for this is that the case of zero velocity is a singular point. It is therefore possible to introduce in updraught conditions a well-defined boundary layer height or top of convection.

It is clear from the examples that we have discussed that the case of a dry atmosphere gives already all the essential features of the problem of the balance of entrainment and buoyancy. Moisture and saturation are additional complications but they do not alter the essence of the solution - again apart from the 'cumulus' case as discussed above. This claim also holds when we add the effects of turbulent diffusion in the entrainment parametrization. We can show this explicitely for the dry atmosphere case. Adding the effects of turbulent diffusion we obtain in stead of (4-5)

$$\begin{aligned} w\frac{\mathrm{d}w}{\mathrm{dz}} &= -\left(\varepsilon|w| + \frac{1}{\tau}\right) w + g\frac{\theta - \bar{\theta}(z)}{\theta(z)}, \\ w\frac{\mathrm{d}\theta}{\mathrm{dz}} &= -\left(\varepsilon|w| + \frac{1}{\tau}\right)\left(\theta - \bar{\theta}(z)\right) \end{aligned}$$

Note that the turbulence parametrization requires one to consider the cases of positive and negative turbulent velocity separately. We will only consider the case of positive velocity and we note that the down-draught case can be obtained by simply replacing ε by $-\varepsilon$. Introducing the usual dimensionless quantities W, T and Z one finds

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -\varepsilon W - 1 + \frac{T}{W}, \tag{39}$$

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -\varepsilon T - \frac{T}{W}. \tag{40}$$

In this case no invariant like the one in Eq. (8) is found. In stead, a differential equation for the quantity $W+T$ is obtained by simply adding (39) and (40). This differential equation can easily be solved allowing the elimination of the vertical velocity from the T-equation. Hence, we find

$$\frac{\mathrm{d}(W+T)}{\mathrm{d}Z} = -\varepsilon(W+T) - 1,$$

For the boundary condition $W(0) = W_0$, $T(0) = T_0$ the solution becomes

$$W + T = \alpha e^{-\varepsilon Z} + \frac{1}{\varepsilon}\left(e^{-\varepsilon Z} - 1\right), \tag{41}$$

where $\alpha = W_0 + T_0$. Elimination of W in Eq. (40) using (41) gives

$$P(Z,T)\mathrm{d}T + Q(Z,T)\mathrm{d}Z = 0,$$

where

$$P(Z,T) = T - \alpha e^{-\varepsilon Z} - \frac{1}{\varepsilon}\left(e^{-\varepsilon Z} - 1\right), \tag{42}$$

and

$$Q(Z,T) = T\left[\varepsilon\left(T - \alpha e^{-\varepsilon Z}\right) - e^{-\varepsilon Z}\right].$$

Note that in the limit $\varepsilon \to 0$ we find that $P \to T + Z - \alpha$ and $Q \to -T$ so that we rediscover the dry problem (9). The effects of turbulent diffusion is a mere change of the height scale.

By multiplying Eq. (42) with the integrating factor $\mu = 1/T^{-2}$ one can show that the resulting equation is exact. The general exact solution is given in (A13) and with the present choice of P and Q, the auxiliary function $\phi(Z) = \varepsilon Z$, so that the general solution becomes

$$\int \mathrm{d}T\, \frac{P(Z,T)}{T^2} + \varepsilon Z + \mathrm{const} = 0.$$

The integral over P can be performed immediately and the resulting exact solution becomes

$$T \log \beta T + \varepsilon Z T = \frac{1}{\varepsilon}\left(1 - e^{-\varepsilon Z}\right) - \alpha e^{-\varepsilon Z} \tag{43}$$

which for small ε obviously has a close relation to the solution for the dry atmosphere case in the absence of turbulent diffusion, by noting that the height scale $Z - \alpha$ simply changes to the right-hand side of (43). The effect of turbulent diffusion on the T_+ type solution for the dry atmosphere is illustrated in Fig. 8. Although there are quantitative differences, e.g. the boundary layer becomes more shallow, it is clear that the essential nature of the solution remains.

Note that in order to study the dominant contribution of the relaxation term in the entrainment parametrization we have studied extensively the problem what happens when the effects of the relaxation term vanishes. Some details are presented in Appendix C.

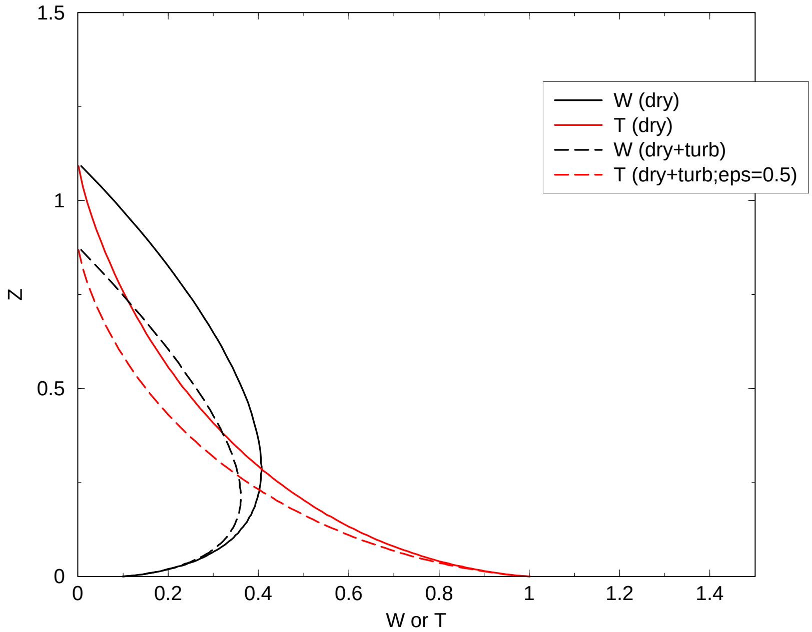

Figure 8: Effect of turbulent diffusion on the T_+ type solution for updraught (full lines, dry; dashed line, dry+turb) for $\varepsilon = 0.5$ The temperature difference at the surface $T_0 = 1$ while the vertical velocity $W_0 = 0.1$.

7 Summary of conclusions

We have presented the exact solution of the updraught equations in the relaxation approximation and we discussed the detailed cases of a dry atmosphere, a moist atmosphere and a cloudy atmosphere. In general the buoyancy term will have the form given in Eq. (29), which we reproduce here for convenience

$$B(T,Q) = a + bT + cQ + dQT + eQ^2,$$

where in the dry case there are only terms linear in T and Q, while in the presence of moisture and cloudiness there are additional nonlinear terms and possibly a constant term. In Appendix B it is shown for a typical case that the size of the nonlinear terms is very small but for the cumulus case the constant term may be considerable. Therefore, apart from the cumulus case, in good approximation the buoyancy term may be written as

$$B(T,Q) = bT + cQ,$$

which means that the updraught solutions have the structure of the dry solution (but, of course, the coefficients may depend on the moisture and cloudiness properties). Therefore, Fig. 1 displays the essence of the updraught solution.

It is emphasized that the exact solution only holds for constant background profiles. We have shown how one can deal with the case of non-uniform profiles. We have shown that if the background profiles are approximated by jump profiles (as is common in numerical models) an exact solution can be found as well. Inside the layers the solution assumes the same form as in the constant background case, but the integration constants are different from one layer to the next. Integrating the updraught equations over the jump, connection formulae for the integration constants have been obtained, so that the solution is completely specified. In particular, we studied the behaviour of the nonlinear solution near an inversion and under certain circumstances (which still have to be obtained in analytical form) an interesting cat's-eye pattern in the solution for the vertical velocity profile is found, which suggests a parametrisation of momentum and heat transfer near the inversion.

Preliminary work has been performed to use this exact solution of the entraining plume equations with $\varepsilon = 1/w\tau$ in the PBL parametrization within the ECMWF single column model cy31r1. The results are very encouraging and allow realistically growing boundary layers with little sensitivity to the parcel entrainment time-scale as expected.

Acknowledgement.

We thank Roel Neggers for his insight in the parameterization of parcel entrainment and Anton Beljaars for his support.

A General solution of updraught equations.

Let us consider the following general updraught equations:

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -1 + \frac{T}{W}F(T), \tag{A1}$$

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -\frac{T}{W}, \tag{A2}$$

where $F(T)$ is an arbitrary function of temperature. We use Eq. (A2) to eliminate $1/W$ from (A1) with the result

$$\frac{\mathrm{d}W}{\mathrm{d}Z} + 1 + F(T)\frac{\mathrm{d}T}{\mathrm{d}Z} = 0. \tag{A3}$$

This can be integrated immediately so that we find the 'invariant'

$$W + Z + \int \mathrm{d}T\, F(T) = \alpha. \tag{A4}$$

We use the invariant to eliminate W from Eq. (A2) to obtain an ordinary, first-order differential equation for the temperature T,

$$P(Z,T)\mathrm{d}T + Q(T)\mathrm{d}Z = 0, \tag{A5}$$

where

$$P = Z - \alpha + \int \mathrm{d}T\, F(T), \text{ and, } Q = -T. \tag{A6}$$

A general treatment to solve equations such as Eqn. (A5) is given by Davis (1962). Suppose that the general solution can be written as

$$\mathscr{H}(Z,T) = 0. \tag{A7}$$

The corresponding differential equation is then

$$\frac{\partial \mathscr{H}}{\partial T}\mathrm{d}T + \frac{\partial \mathscr{H}}{\partial Z}\mathrm{d}Z = 0. \tag{A8}$$

Comparing (A5) with (A8) and observing that

$$\frac{\partial^2 \mathscr{H}}{\partial Z \partial T} = \frac{\partial^2 \mathscr{H}}{\partial T \partial Z} \tag{A9}$$

we see that (A5) is immediately integrable provided P and Q satisfy the following condition

$$\frac{\partial P}{\partial Z} = \frac{\partial Q}{\partial T}. \tag{A10}$$

In this case it is said that Eq. (A5) is *exact* and its solution is obtain by writing: $\partial\mathscr{H}/\partial T = P$ and $\partial\mathscr{H}/\partial Z = Q$. Integrating the first equation one finds

$$\mathscr{H} = \int \mathrm{d}T\ P(Z,T) + \phi(Z), \tag{A11}$$

and the unknown function ϕ is obtained by substituting (A11) in $\partial\mathscr{H}/\partial Z = Q$ with the result

$$\phi(Z) = \int \mathrm{d}Z \left\{ Q(Z,T) - \frac{\partial}{\partial Z}\int \mathrm{d}T\ P(Z,T) \right\}. \tag{A12}$$

When the solvability condition (A10) is used in the curly bracketed term of (A12) one would be inclined to conclude that the curly bracketed term vanishes, but this is only true apart from a constant. This is easily seen because if Q satisfies (A10) then $Q+\mathrm{const}$ satisfies the solvability condition as well. Hence, in general ϕ does not need to vanish.

The general solution becomes

$$\mathscr{H} = \int \mathrm{d}T\ P(Z,T) + \phi(Z) + \mathrm{const} = 0. \tag{A13}$$

Unfortunately, for the present case Eq. (A5) is not exact as $\partial P/\partial Z = 1$, while $\partial Q/\partial T = -1$. If Eq. (A5) is not exact, it is theoretically possible, and in some examples it is practical, to make the equation exact by introducing as a multiplier a so-called *integrating factor* $\mu(Z,T)$. This function must satisfy the equation

$$\frac{\partial \mu P}{\partial Z} = \frac{\partial \mu Q}{\partial T}, \tag{A14}$$

or

$$Q\frac{\partial \mu}{\partial T} - P\frac{\partial \mu}{\partial Z} + \mu\left(\frac{\partial Q}{\partial T} - \frac{\partial P}{\partial Z}\right) = 0. \tag{A15}$$

In general this partial differential equation for μ is more difficult to solve than the original problem, unless it is possible that a special choice for μ satisfies the equation. In this case we are lucky because the choice that the integrating factor is only a function of T, hence $\mu = \mu(T)$, works. For this choice of μ Eq. (A15) may be solved with the result $\mu = 1/T^2$. Furthermore, for this particular choice of P and Q it can be shown that ϕ vanishes, and the general solution of (A5) becomes

$$\mathscr{H} = \int \mathrm{d}T \frac{P(Z,T)}{T^2} + \mathrm{const} = 0. \tag{A16}$$

Making use of the expression for P in (A6) we obtain

$$\mathscr{H} = \int \frac{\mathrm{d}T}{T^2}\left(Z - \alpha + \int^T \mathrm{d}T' F(T')\right) + \mathrm{const} = 0. \tag{A17}$$

The integration over the first term can be performed immediately and we obtain the main result

$$-\frac{Z-\alpha}{T} + \int \frac{\mathrm{d}T}{T^2}\int^T \mathrm{d}T' F(T') = \mathrm{const}. \tag{A18}$$

Let us consider the case of a power expansion of F in terms of T,

$$F(T) = \frac{a_{-1}}{T} + \sum_{n=0}^{\infty} a_n T^n, \tag{A19}$$

where we have isolated a term proportional to $1/T$, hence the buoyancy term $T f(T)$ may remain finite for vanishing T (this case may arise when the parcel is cloudy while the environment is clear). It is now straightforward to evaluate the double integral in (A18) and we obtain the solution

$$Z - \alpha = -a_{-1}(1 + \log T) + a_0 T \log(\beta T) + \sum_{n=1}^{\infty} \frac{a_n}{n(n+1)} T^{n+1}. \tag{A20}$$

where β is the second integration coefficient which follows by writing const $= a_0 \log \beta$.

The dry atmosphere case now follows by letting all expansion coefficients a_n to vanish, except the first one which is taken equal to one. Hence for the dry atmosphere we find

$$Z - \alpha = T \log(\beta T). \tag{A21}$$

Furthermore, the moist, non-cloudy case follows by choosing $a_0 = 1 + a\gamma$, and $a_1 = a\gamma$, hence the solution becomes

$$Z - \alpha = (1 + a\gamma) T \log(\beta T) + \frac{a\gamma}{2} T^2. \tag{A22}$$

Finally, the cumulus case follows by choosing finite coefficients for $a_0 = A$, $a_1 = B$ and $a_{-1} = C$, while all other coefficients vanish. The solution becomes

$$Z - \alpha = -C(1 + \log T) + AT \log(\beta T) + \frac{B}{2} T^2. \tag{A23}$$

Therefore, it is concluded that as long as the buoyancy term can be written as a function of temperature, an exact solution of the updraught equations for a parcel may be obtained.

B Buoyancy term for cloudy situations as function of q_t and θ_l

In the vertical velocity equation (24) the buoyancy term B is involved because the virtual potential temperature depends on the saturation mixing ratio q_{sat} (23), which itself depends on temperature. The derivation requires the following standard equations.

$$\theta = T \left(\frac{p_S}{p}\right)^{R/c_p} \equiv T\,\Pi(p) \tag{B1}$$

$$\theta_v = \theta(1 + 0.61q - q_l) \tag{B2}$$

$$\theta_l = \theta - \frac{L}{c_p} \Pi(p) q_l \tag{B3}$$

$$q_t = q + q_l \tag{B4}$$

For clear situations with $q_l = 0$ and $\theta_l = \theta$ the virtual potential temperature $\theta_{v,clear}$ as a function of conserved variables θ_l and q_t simply reduces to

$$\theta_{v,clear} = \theta_l(1 + 0.61q_t). \tag{B5}$$

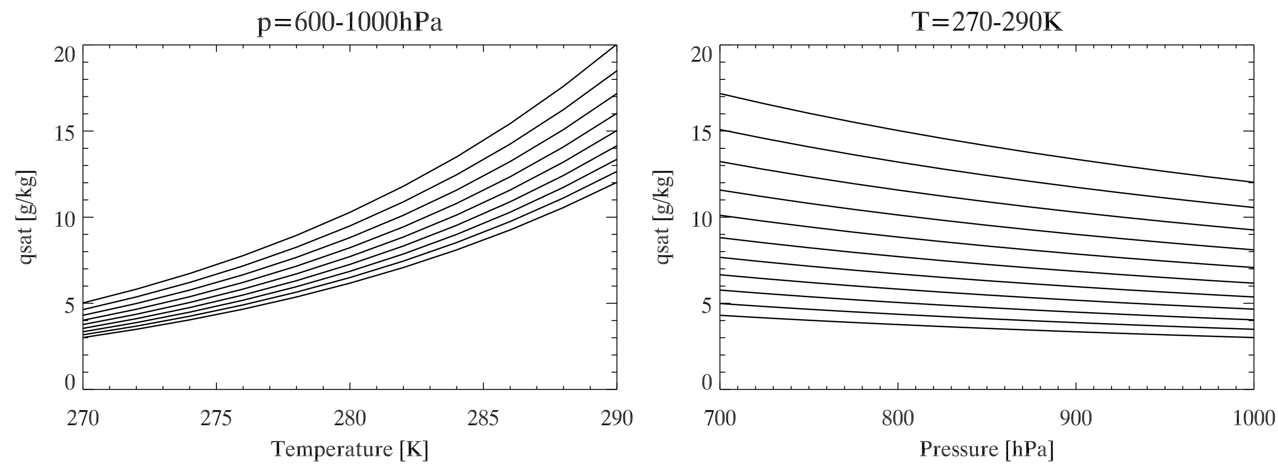

Figure B1: q_{sat} as a function of T (left panel) and as a function of p (right panel). The top line corresponds to p=600hPa and the lowest to p=1000hPa (left) and T=290K and T=270K (right).

For cloudy situations T and q are linked through the condensation process. We will solve this interaction by applying a linearisation in q_{sat} and by assuming that the effects of the condensation on the temperature are relatively small. First we write formally T as function of conserved variables θ_l, q_t and non-conservative q_{sat}, where it is assumed that all supersaturation condenses into liquid water:

$$T = \frac{\theta_l}{\Pi(p)} + \frac{L}{c_p}(q_t - q_{sat}) \tag{B6}$$

By using the expression for q_{sat} (Eq. 23) in Eq. (B6) it is straightforward to solve for the temperature T by means of Newton iteration. However, a simple and accurate solution can also be found by linearising q_{sat} in T around an appropriately chosen reference temperature T_0. Here,

$$\left.\frac{\partial q_{sat}}{\partial T}\right|_p = \frac{R_{dry}}{R_{vap}}\frac{AB}{pT^2}e^{-B/T} = \frac{B}{T^2}q_{sat} \tag{B7}$$

and to a good approximation (cf. Fig. B1) we have

$$q_{sat}(T) = q_{sat}(T_0) + \left.\frac{\partial q_{sat}}{\partial T}\right|_p (T - T_0) \tag{B8}$$

Assuming a non-condensation reference state ($q_{l,0} = 0$), temperature T_0 follows from Eq. (B6) as

$$T_0 = \frac{\theta_l}{\Pi(p)}. \tag{B9}$$

Substitution of Eq. (B8) into B6 gives an equation for the temperature T which can be written as

$$T = T_0 + \frac{L}{c_p}\frac{q_t - q_{sat}(T_0)}{1+\gamma_0}, \tag{B10}$$

where

$$\gamma_0 \equiv \frac{L}{c_p}\left.\frac{\partial q_{sat}}{\partial T}\right|_{p,T_0}. \tag{B11}$$

Substitution of Eq. (B10) back into (B8) gives for q_{sat} the result

$$q_{sat}(T) = \frac{q_{sat}(T_0)+\gamma_0 q_t}{1+\gamma_0} \tag{B12}$$

Results from Eq. (B12) and (B10) were found to be remarkably close to the results from solving (B6) by means of Newton iteration.

The virtual potential temperature $\theta_{v,cloud}$ can be written as

$$\theta_{v,cloud} = \left(\theta_l + \frac{L}{c_p}\Pi(p)(q_t - q_{sat})\right)(1+1.61q_{sat} - q_t) \tag{B13}$$

and eliminating q_{sat} by means of (B12) the virtual potential temperature can be written in terms of conserved variables

$$\theta_{v,cloud}(q_t,\theta_l) = \left\{\theta_l + \frac{L}{c_p}\Pi(p)\left(q_t - \frac{q_{sat}(T)+\gamma q_t}{1+\gamma}\right)\right\}\left(1+1.61\frac{q_{sat}(T)+\gamma q_t}{1+\gamma} - q_t\right), \tag{B14}$$

or, written explicitely

$$\theta_{v,cloud}(q_t,\theta_l) = \left(\frac{1}{1+\gamma}\right)^2 (a + b\,\theta_l + c\,q_t + d\,q_t\theta_l + e\,q_t^2) \tag{B15}$$

with

$$\begin{aligned}
a &= -\frac{L}{c_p}\Pi(p)\,(1+\gamma+1.61q_{sat}(T))\;q_{sat}(T) && \approx -44.5K\\
b &= (1+\gamma)\,(1+\gamma+1.61q_{sat}(T)) && \approx 5.25\\
c &= [1+\gamma+(2.61-0.61\gamma)q_{sat}(T)]\;\frac{L}{c_p}\Pi(p) && \approx 5890K\\
d &= (1+\gamma)\,(0.61\gamma-1) && \approx -0.494\\
e &= \frac{L}{c_p}\Pi(p)\,(0.61\gamma-1) && \approx -553K
\end{aligned}$$

Approximate values for the constants a to e were added assuming $L = 2.5\cdot 10^6 J/kg$, $c_p = 1005J/kgK$, p=900hPa, θ_l=290K, Π=1.03, $q_{sat}(T)$ = 7.55 g/kg, $\gamma = 1.29$ K^{-1} and $1/(1+\gamma)^2 \approx 0.191$. For convenience $q_{sat}(T_0)$ and γ_0 were written as $q_{sat}(T)$ and γ respectively.

To determine the buoyancy B one needs to distinguish between cloudy and non-cloudy environments such as encountered in stratocumulus and cumulus respectively. This is because the functional form of the environmental virtual potential temperature $\overline{\theta_v}$ is Eq. (B15) for cloudy environments and Eq. (B5) for clear environments. As will be seen later, having different formulations for θ_v in parcel and environment introduces a constant term in the buoyancy term. This leads to qualitative differences in approaching parcel rest. The behaviour of these θ_v's as a function of conserved variables θ_l and q_t is illustrated in Fig. B2 for clear and cloudy cases respectively. Both cases can be combined in one figure (4) for the regime below and above the saturation line respectively. Here, the red line connects all states that are neutrally buoyant with respect to the environment. All states above that line are therefore buoyant. Three typical cases can now be distinguished:

(1) the dry PBL or sub-cloud layer with a dry parcel and a dry environment,

(2) the stratocumulus with a cloudy parcel and a cloudy environment, and

(3) the cumulus with a cloudy parcel and a clear environment.

Case (1) and (2) are rather similar because the formulation of buoyancy is identical between parcel and environment. Therefore, the parcel continually approaches the environment through entrainment in terms of conserved variables as well as buoyancy until it comes to rest. Case (3) - the cumulus case - is rather different because of the difference of formulation in θ_v between parcel and environment. As can be seen in figure 4, the entrainment process towards the environment involves three stages: (3a) The buoyant parcel slowly looses it's buoyant character. (3b) The parcel becomes moderately negatively buoyant but remains cloudy. This stage could lead to creating an anvil and it's length in phase space depends on how cold and dry (!) the original parcel is. And (3c) the dry downdraught.

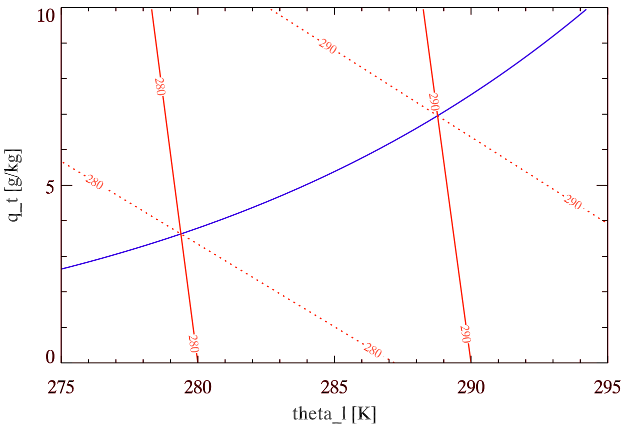

Figure B2: Virtual potential temperature θ_v (red lines) for clear (solid) and cloudy (dotted) conditions as a function of conserved variables. The blue line represents saturation with cloudy conditions above. A pressure of 900hPa is used.

B.1 Cloudy parcel - cloudy environment: stratocumulus

We now write the buoyance term for the stratocumulus case in terms of normalised conserved variables:

$$T \equiv \frac{\theta_l - \overline{\theta}_l}{\theta_l}, \tag{B16}$$

$$Q \equiv \frac{q_t - \overline{q}}{\overline{q}}. \tag{B17}$$

The result is

$$\begin{aligned} B_{strcu}(q_t, \theta_l) &= g \frac{\theta_{v,cloud} - \overline{\theta}_{v,cloud}}{\theta_{v,cloud}} \\ &= \frac{g}{\theta_{v,cloud}(1+\gamma)^2} \left\{ \left(b\overline{\theta}_l + d\,\overline{q}\overline{\theta}_l \right) T + \left(c\,\overline{q} + d\,\overline{q}\overline{\theta}_l + 2e\,\overline{q}^2 \right) Q + d\,\overline{q}\overline{\theta}_l QT + e\,\overline{q}^2 Q^2 \right\} \end{aligned} \tag{B18}$$

Note that this function has only linear and quadratic terms in the perturbations of conserved variables Q and T. Therefore buoyancy vanishes when the parcel perturbations approach zero.

Following the example case started above and using $\overline{\theta}_l = 290$ and $\overline{q} = 10g/kg$ we get explicitely

$$B_{strcu}(q_t, \theta_l) = g\left\{0.991\ T + 0.0374\ Q - 0.000933\ QT - 0.0000361\ Q^2\right\}. \tag{B19}$$

Taking parcel perturbations of T and Q of order 1/300 and 1/10 we conclude that both linear terms are of the same order (~ 0.003), but the quadratic terms are 4 orders of magnitude smaller (3×10^{-7}).

B.2 Cloudy parcel - clear environment: cumulus

The buoyancy for the cumulus case is constructed by using

$$B_{cum}(q_t,\theta_l) = g\frac{\theta_{v,cloud} - \overline{\theta}_{v,clear}}{\overline{\theta}_{v,clear}}$$

The result is

$$\begin{aligned} B_{cum}(q_t,\theta_l) = \frac{g}{\overline{\theta}_{v,clear}(1+\gamma)^2}\Big\{ & a + b\overline{\theta}_l + c\,\overline{q}_t + d\,\overline{q}_t\overline{\theta}_l + e\,\overline{q}_t^2 - \overline{\theta}_l - 0.61\,\overline{q}_t\overline{\theta}_l \\ & + T\left(b\overline{\theta}_l + d\,\overline{q}_t\overline{\theta}_l\right) + Q\left(c\,\overline{q}_t + d\,\overline{q}_t\overline{\theta}_l + 2e\,\overline{q}_t^2\right) + QTd\,\overline{q}_t\overline{\theta}_l + Q^2 e\,\overline{q}_t^2\Big\} \end{aligned} \tag{B21}$$

or

$$\begin{aligned} B_{cum}(q_t,\theta_l) = \frac{g}{\overline{\theta}_{v,clear}(1+\gamma)^2}\Big\{ & \overline{\theta}_{v,cloud} - \overline{\theta}_{v,clear} \\ & + T\left(b\overline{\theta}_l + d\,\overline{q}_t\overline{\theta}_l\right) + Q\left(c\,\overline{q}_t + d\,\overline{q}_t\overline{\theta}_l + 2e\,\overline{q}_t^2\right) + QTd\,\overline{q}_t\overline{\theta}_l + Q^2 e\,\overline{q}_t^2\Big\} \end{aligned} \tag{B22}$$

The constant term in B_{cum} is particularly interesting as it changes the shape of the solution. This term can be written as $\overline{\theta}_{v,cloud} - \overline{\theta}_{v,clear}$, the difference between the cloudy and clear formulations of θ_v given the environmental conditions ($\overline{q}_t$, $\overline{\theta}_l$). This term can be illustrated by drawing the virtual potential temperature as a function of mixing fraction (see Fig. 3). This mixing fraction corresponds to the convective mixing line in Fig. 4 (green line “convection”). Note that this term is always negative and remains constant throughout the entrainment process because it only depends on the environment.

To get an impression of the orders of magnitudes of the individual terms in B_{cum} we use the typical example of cumulus values as in Eq. B19 (see also Fig 3). Eq. (B22) then becomes

$$B_{cum}(q_t,\theta_l) = g\Big\{-0.00142 + 0.998\;T + 0.0376\;Q - 0.000940\;QT - 0.0000363\;Q^2\Big\}. \tag{B23}$$

When taking parcel perturbations of T and Q are of order 1/300 and 1/10 we see for the cumulus case that the unique constant term is around -0.001, both linear terms are of the same order ($\sim$ 0.003), but the quadratic terms are 4 orders of magnitude smaller (3×10^{-7}). Interestingly, the buoyancy in the cumulus case differs from the stratocumulus case purely due to the first constant term, that describes the difference between virtual potential temperature formulations in clear and cloudy conditions.

C Limit of Eqns. (39-40).

We wish to study the limiting behaviour of the set of Eqns. (39-40) for small effects of the relaxation term. In order to do so, consider the boundary value problem

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -\varepsilon W - \delta + \frac{T}{W}, \tag{C1}$$

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -\varepsilon T - \frac{\delta T}{W}, \tag{C2}$$

subject to the boundary conditions

$$T(0) = T_0,\; W(0) = W_0. \tag{C3}$$

Here, δ measures the strength of the relaxation term. Let us first consider the case of vanishing δ from the start. Using the boundary condition, one then finds immediately for T

$$T(Z) = T_0 e^{-\varepsilon Z}. \tag{C4}$$

This is substituted in Eq. (C1) which we write as an equation for the 'energy' $E = W^2/2$ because in terms of E the problem becomes linear. We find for E

$$E(Z) = \frac{2T_0}{\varepsilon} e^{-\varepsilon Z} + \left(E_0 - \frac{2T_0}{\varepsilon}\right) e^{-2\varepsilon Z}, \tag{C5}$$

where $E_0 = W_0^2/2$.

Let us now try to obtain this limiting behaviour from the exact solution to Eqns. (C1-C2). By introduction of new dependent variables

$$T' = \frac{T}{\delta^2},\ W' = \frac{W}{\delta}, \tag{C6}$$

one obtains the set of equations

$$\frac{\mathrm{d}W'}{\mathrm{d}Z} = -\varepsilon W' - 1 + \frac{T'}{W'}, \tag{C7}$$

$$\frac{\mathrm{d}T'}{\mathrm{d}Z} = -\varepsilon T' - \frac{T'}{W'}, \tag{C8}$$

which are identical to (39)-(40). Therefore, the exact solution becomes in terms of the original variables

$$T \log \beta T + \varepsilon Z T = \frac{\delta^2}{\varepsilon}\left(1 - e^{-\varepsilon Z}\right) - (T_0 + \delta w_0) e^{-\varepsilon Z}, \tag{C9}$$

where

$$\beta = \frac{1}{T_0} e^{-1-\delta w_0/T_0}. \tag{C10}$$

Elimination of β in (C9) and rearrangement of the terms gives

$$T\left(1 - \log\frac{T}{T_0} - \varepsilon Z\right) - T_0 e^{-\varepsilon Z} = \delta W_0 \left(e^{-\varepsilon Z} - \frac{T}{T_0}\right) - \frac{\delta^2}{\varepsilon}\left(1 - e^{-\varepsilon Z}\right). \tag{C11}$$

In the limit of vanishing δ the left-hand side of Eq. (C11) vanishes. It turns out that this happens for $T = T_0 \exp(-\varepsilon Z)$. In fact, for this choice of temperature profile, the term proportional to δ vanishes as well, hence from (C11) we find

$$T = T_0 e^{-\varepsilon Z} + \mathscr{O}(\delta^2). \tag{C12}$$

Using (C12), the vertical velocity profile then follows from integrating (C1) in the limit of $\delta \to 0$ and the result is, of course, identical to (C5).

On the other hand, if one takes the limit of vanishing ε in Eq. (C9) one rediscovers for $\delta = 1$ the exact solution for the dry problem, Eq. (10). Therefore, we have checked that the exact solution (43) has the correct limiting behaviour for small ε and small δ, giving confidence in the correctness of the solution.

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